A Queueing system with Catastrophe, Feedback, State dependent Service and Environmental change

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Abstract: In this paper, a finite capacity queueing system with state dependent service operating in different environments with catastrophes and feedback is studied. The service rate increases (decreases) according as n, the number of units in the system, is less (greater) than N, a pre-assigned number. We undertake the transient analysis of a limited capacity queueing system with two environmental states in the presence of catastrophes and feedback. Transient state solution is obtained by using the technique of probability generating function. The steady state results of the model is obtained by using the property of Laplace transform. Finally, some particular cases of the queueing model are also derived and discussed.

Keywords: Catastrophes, Feedback, Environment, Service rate, Probability generating function, Laplace transform.

1. **Introduction:**

In the recent years, many researchers working in the field of queueing theory focused on the M/M/1 queueing systems that include the effect of catastrophes and feedback. Whenever a catastrophe occurs at the system, all the customers there are destroyed immediately, the server gets inactivated momentarily, and the server is ready for service when a new arrival occurs. Feedback in queueing literature represents customer dissatisfaction because of inappropriate quality of service. In case of feedback, after getting partial or incomplete service, customer retries for service. Also, we have another factor of environmental change, i.e. the change in the environment affects the state of the queueing system. In other words, the state of the queueing system is a function of environmental change factors.

This paper is the generalization of our previous work [Kumar, Darvinder (9)] in which we do not consider the concept of customers feedback. Previously we assumed that the served customer leaves the system permanently and never return back. In this paper we consider that the served customer may return back for getting service if it is not satisfied by their previous service given by the server. After obtaining service, if the customers are unsatisfied (feedback), they will return back to the service terminal as a new arrival or they will leave the service station permanently as satisfied customers. The production firm may for various seasons stop the supply of the product at some time which make them zero instantaneously and start it again can be regarded as the occurrence of a

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catastrophe. Therefore, the proposed model is highly useful for dealing with real-world queuing situations such as manufacturing firms or banking sector etc.

Many authors have used the concept of effect of environmental changing states, feedback and catastrophes in queues see e.g. Jain and Kanethia [7], Bura, Gulab et. al [3], Goel, L.R. [6]. In [15] Thangaraj and Vanitha, obtained transient solution of M/M/1 feedback queue with catastrophes using continued fractions and the steady-state solution. Kalidass et al. [8] studied the time dependent analysis of an M/M/1/N queue with catastrophes and a repairable server. Kumar and Sharma [12] studied a M/M/1 Feedback queuing model with retention of reneged customers and Balking.
The layout of this paper is as follows. In the next section we present the assumptions and definitions of the queueing model. Section 3 provides a detailed analysis of the main model, which is used in section 4 in proving some particular cases. Steady state results are also derived and discussed along with the application of the model in section 5 and 6.

2. Assumptions and Definitions:

(i) The customers arrive in the system one by one in accordance with a Poisson process at a single service station. The arrival pattern is non-homogeneous, i.e. there may exist two arrival rates, namely λ_1 and 0 of which only one is operative at any instant.

(ii) The customers are served one by one at the single channel. The service time is exponentially distributed. Further, it has been assumed that corresponding to arrival rate λ_1 the Poisson service rate is a_n and the service rate corresponding to the arrival rate 0 is b_n. The state of the queueing system when operating with arrival rate λ_1 and service rate a_n is designated as E whereas the other with arrival rate 0 and service rate b_n is designated as F.

(iii) The Poisson service rate a_n is assumed to depend on the number waiting in the queue, including the one in service in such a manner that whenever this number (say n) is equal to some fixed number (say N), we have some normal rate as μ_1 and for number of units greater than N, the rate is higher and for number of units less than N it is lower than the normal rate. We therefore, define

$$a_n = \mu_1[1+\varepsilon(n-N)] \quad \text{with} \quad n \geq N - \frac{1}{\varepsilon}$$

$$\text{and} \quad 0 \leq N - \frac{1}{\varepsilon} \leq n \leq M$$

Where M denotes the size of the waiting space and ε is a positive number $\geq \frac{1}{N}$. This restriction on M is necessary to avoid a negative value of a_n. Similarly, the Poisson service rate b_n is defined as

$$b_n = \mu_2[1+\varepsilon(n-N)] \quad \text{with} \quad n \geq N - \frac{1}{\varepsilon}$$

$$\text{and} \quad 0 \leq N - \frac{1}{\varepsilon} \leq n \leq M$$

(iv) In state E after obtaining a service, an unsatisfied customer may rejoin the queue for receiving another service with probability 1-p (=q), referred to as "feedback" or they can choose to leave the system permanently with probability p, p+q=1. There is no feedback in state F of the system.

(v) The Poisson rates at which the system moves from environmental states F to E and E to F are denoted by α and β respectively.

(vi) When the system is not empty, catastrophes occur according to a Poisson process with rate ξ. The effect of each catastrophe is to make the queue instantly empty. Simultaneously, the system becomes ready to accept the new customers.

(vii) The queue discipline is first- come- first- served.

(viii) The capacity of the queueing system is limited to M. i.e., if at any instant there are M units in the queue then the units arriving at that instant will not be permitted to join the queue, it will be considered lost for the system.

3. Formulation of Model and Analysis (Time Dependent Solution):

Define,

$P_n(t)$ = Joint probability that at time t the system is in state E and n units are in the queue, including the one in service.

$Q_n(t)$ = Joint probability that at time t the system is in state F and n units are in the queue, including the one in service.

$R_n(t)$ = The probability that at time t there are n units in the queue, including the one in service.

Obviously,

$$R_n(t) = P_n(t) + Q_n(t)$$

Let us reckon time t from an instant when there are zero customers in the queue and the system is in the environmental state E so that the initial conditions associated with $P_n(t)$ and $Q_n(t)$ becomes,

$$P_n(0) = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$Q_n(0) = 0 \; ; \quad \text{for all } n.$$

The differential -difference equations governing the system are:

$$\frac{d}{dt}P_0(t) = -(\lambda_1+\beta+\xi)P_0(t) + a_1 p P_1(t) + \alpha Q_0(t) + \xi\sum_{n=0}^{M} P_n(t)\; ; \; n = 0 \qquad \text{..... (1)}$$

$$\frac{d}{dt}P_n(t) = -(\lambda_1 + a_n p + \beta + \xi)P_n(t) + a_{n+1} p P_{n+1}(t) + \lambda_1 P_{n-1}(t) + \alpha Q_n(t)\; ; \; 0 < n < M \qquad \text{.... (2)}$$

$$\frac{d}{dt}P_M(t) = -(a_M p + \beta + \xi)P_M(t) + \lambda_1 P_{M-1}(t) + \alpha Q_M(t)\; ; \; n = M \qquad \text{.... (3)}$$

$$\frac{d}{dt}Q_0(t) = -(\alpha+\xi)Q_0(t) + b_1 Q_1(t) + \beta P_0(t) + \xi\sum_{n=0}^{M} Q_n(t)\; ; \quad n = 0 \qquad \text{.... (4)}$$

$$\frac{d}{dt}Q_n(t) = -(b_n + \alpha + \xi)Q_n(t) + b_{n+1} Q_{n+1}(t) + \beta P_n(t)\; ; \; 0 < n < M \qquad \text{..... (5)}$$

$$\frac{d}{dt}Q_M(t)=-(b_M+\alpha+\xi)Q_M(t)+\beta P_M(t);\quad n=M \qquad \text{..... (6)}$$

Define, the Laplace Transform as

$$\text{L.T. } [f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt = \bar{f}(s) \qquad \text{.....(7)}$$

Now, taking the Laplace transforms of equations (1)–(6) and using the initial conditions, we get

$$(s+\lambda_1+\beta+\xi)\overline{P}_0(s)-1=a_1 p\overline{P}_1(s)+\alpha\,\overline{Q}_0(s)+\xi\sum_{n=0}^{M}\overline{P}_n(s) \qquad \text{`.... (8)}$$

$$(s+\lambda_1+a_n p+\beta+\xi)\overline{P}_n(s)=a_{n+1}\,p\overline{P}_{n+1}(s)+\lambda_1\,\overline{P}_{n-1}(s)+\alpha\,\overline{Q}_n(t) \qquad \text{.... (9)}$$

$$(s+a_M p+\beta+\xi)\overline{P}_M(s)=\lambda_1\,\overline{P}_{M-1}(s)+\alpha\,\overline{Q}_M(s) \qquad \text{... (10)}$$

$$(s+\alpha+\xi)\overline{Q}_0(s) = b_1\,\overline{Q}_1(s)+\beta\,\overline{P}_0(s)+\xi\sum_{n=0}^{M}\overline{Q}_n(s) \qquad \text{.... (11)}$$

$$(s+b_n+\alpha+\xi)\overline{Q}_n(s) = b_{n+1}\,\overline{Q}_{n+1}(s)+\beta\,\overline{P}_n(s) \qquad \text{.... (12)}$$

$$(s+b_M+\alpha+\xi)\overline{Q}_M(s)=\beta\,\overline{P}_M(s) \qquad \text{.... (13)}$$

Define, the probability generating functions

$$P(z,s)=\sum_{n=0}^{M}\overline{P}_n(s)z^n \qquad \text{.... (14)}$$

$$Q(z,s)=\sum_{n=0}^{M}\overline{Q}_n(s)z^n \qquad \text{.... (15)}$$

$$R(z,s)=\sum_{n=0}^{M}\overline{R}_n(s)z^n \qquad \text{.... (16)}$$

where

$$\overline{R}_n(s)=\overline{P}_n(s)+\overline{Q}_n(s)$$

Multiplying equations (8)–(10) by the suitable powers of z, summing over all n and using equations (14)–(16), we have.

$$z(z-1)\mu_1 p\varepsilon P'(z,s)+\left[zs+\mu_1 p(z-1)(1-\varepsilon N)+\lambda_1 z(1-z)+\beta z+\xi z\right]P(z,s)-\alpha\, zQ(z,s)$$

$$=z+\mu_1 p(1-\varepsilon N)(z-1)\overline{P}_0(s)+\lambda_1\, z^{M+1}(1-z)\overline{P}_M(s)+\xi\, z\sum_{n=0}^{M}\overline{P}_n(s) \qquad \text{....}$$

(17)

Similarly, from equations (11)–(13) and using equations (14)–(16), we have

$$z(z-1)\,\mu_2\varepsilon Q'(z,s)+[zs+\mu_2(z-1)(1-\varepsilon N)+\alpha z+\xi z]\,Q(z,s)-\beta zP(z,s)$$

$$=\mu_2(1-\varepsilon N)(z-1)\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s) \qquad \text{... (18)}$$

In order to obtain P(z,s) and Q(z,s) from equation (17) & (18), we use **Iteration Method.** If we assume that the parameter β is small, then we can use it in the series solution as follows:

$$P(z,s)=P_0(z,s)+\beta P_1(z,s)+\cdots\cdots\cdots \quad \text{..... (19)}$$

$$Q(z,s)=Q_0(z,s)+\beta Q_1(z,s)+\cdots\cdots\cdots \quad \text{..... (20)}$$

Where the non-written terms are of higher order of β (i.e., we limit ourselves to the first approximation). Substituting values of P(z,s) and Q(z,s) from equations (19) and (20) in equations (17) and (18) and identifying terms with like powers of β. We obtain thus the zero order (i.e., terms not containing β) and one order (i.e., terms containing first power of β) approximations:

$$P_0'(z,s)+\eta_1(z)\,P_0(z,s)-\frac{\alpha}{\mu_1 p\varepsilon\varepsilon(-1)}Q_0(z,s)=z_1 \quad \text{.... (21)}$$

$$Q_0'(z,s)+\eta_2(z)\,Q_0(z,s)=z_2 \quad \text{.... (22)}$$

$$P_1'(z,s)+\frac{1}{(z-1)\mu_1 p\varepsilon}P_0(z,s)+\eta_1(z)\,P_1(z,s)-\frac{\alpha}{(z-1)\mu_1 p\varepsilon}Q_1(z,s)=0 \quad \text{.... (23)}$$

$$Q_1'(z,s)+\eta_2(z)\,Q_1(z,s)-\frac{1}{(z-1)\mu_2\varepsilon}P_0(z,s)=0 \quad \text{... (24)}$$

where,

$$\eta_1(z)=\frac{zs+\mu_1 p(z-1)(1-\varepsilon N)+\lambda_1 z(1-z)+\xi z}{z(z-1)\mu_1 p\,\varepsilon}$$

$$\eta_2(z)=\frac{zs+\mu_2(z-1)(1-\varepsilon N)+\alpha z+\xi z}{z(z-1)\mu_2\,\varepsilon}$$

$$z_1=\frac{z+\mu_1 p(1-\varepsilon N)(z-1)\overline{P}_0(s)+\lambda_1 z^{M+1}(1-z)\overline{P}_M(s)+\xi z\sum_{n=0}^{M}\overline{P}_n(s)}{z(z-1)\mu_1 p\,\varepsilon}$$

$$z_2=\frac{\mu_2(1-\varepsilon N)(z-1)\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s)}{z(z-1)\mu_2\,\varepsilon}$$

On solving equation (22), we have

$$Q_0(z,s)=\frac{M(z)\ \overline{Q}_0(s)+N(z)\ \sum_{n=0}^{M}\overline{Q}_n(s)}{A(z)} \quad \text{... (25)}$$

where

$$A(z)=z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(s+\alpha+\xi)}{\mu_2\varepsilon}}$$

$$M(z)=\frac{1-\varepsilon N}{\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}-1}\cdot(z-1)^{\frac{(s+\alpha+\xi)}{\mu_2\varepsilon}}\,dz$$

$$N(z)=\frac{\xi}{\mu_2\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(s+\alpha+\xi)}{\mu_2\varepsilon}-1}\,dz$$

Solving equations (25) and (21) for $P_0(z,s)$, we have

$$P_0(z,s)=\frac{\overline{Q}_0(s)K_1(z)+\overline{P}_0(s)K_2(z)+\overline{P}_M(s)K_3(z)+\sum_{n=0}^{M}\overline{Q}_n(s)K_4(z)+\sum_{n=0}^{M}\overline{P}_n(s)K_5(z)+K_6(z)}{B(z)} \quad \ldots (26)$$

where

$$B(z)=z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(s+\xi)}{\mu_1 p\varepsilon}}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}$$

$$K_1(z)=\frac{\alpha}{\mu_1 p\varepsilon}\int_0^z (z-1)^{\frac{s+\xi}{\mu_1 p\varepsilon}-\left\{\frac{(s+\alpha+\xi)}{\mu_2\varepsilon}+1\right\}}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}\cdot M(z)\,dz$$

$$K_2(z)=\frac{1-\varepsilon N}{\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}-1}\cdot(z-1)^{\frac{(s+\xi)}{\mu_1 p\varepsilon}}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}\,dz$$

$$K_3(z)=-\frac{\lambda_1}{\mu_1 p\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}+M}\cdot(z-1)^{\frac{(s+\xi)}{\mu_1 p\varepsilon}}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}\,dz$$

$$K_4(z)=\frac{\alpha}{\mu_1 p\varepsilon}\int_0^z (z-1)^{\frac{s+\xi}{\mu_1 p\varepsilon}-\left\{\frac{(s+\alpha+\xi)}{\mu_2\varepsilon}+1\right\}}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}\cdot N(z)\,dz$$

$$K_5(z)=\frac{\xi}{\mu_1 p\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{s+\xi}{\mu_1 p\varepsilon}-1}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}\,dz$$

$$K_6(z)=\frac{1}{\mu_1 p\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(s+\xi)}{\mu_1 p\varepsilon}-1}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}\,dz$$

Solving equations (26) and (24) for $Q_1(z,s)$, we have

$$Q_1(z,s)=\frac{\overline{Q}_0(s)K_7(z)+\overline{P}_0(s)K_8(z)+\overline{P}_M(s)K_9(z)+\sum_{n=0}^{M}\overline{Q}_n(s)K_{10}(z)+\sum_{n=0}^{M}\overline{P}_n(s)K_{11}(z)+K_{12}(z)}{A(z)}$$

….. (27)

where,

$$K_{i+6}(z)=\frac{1}{\mu_2\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(s+\alpha+\xi)}{\mu_2\varepsilon}-1}\cdot K_i(z)\,dz \qquad ;\ i=1,2,3,4,5,6.$$

Again solving equations (27), (26) and (23) for $P_1(z,s)$, we have

$$P_1(z,s)=\frac{\overline{Q}_0(s)K_{13}(z)+\overline{P}_0(s)K_{14}(z)+\overline{P}_M(s)K_{15}(z)+\sum_{n=0}^{M}\overline{Q}_n(s)K_{16}(z)+\sum_{n=0}^{M}\overline{P}_n(s)K_{17}(z)+K_{18}(z)}{B(z)}$$

….. (28)

where

$$K_{i+j+(6-j)}(z)=\frac{1}{\mu_1\varepsilon}\int_0^z\frac{1}{z-1}\left[\frac{\alpha B(z)}{A(z)}K_i(z)-K_{j+1}(z)\right]dz \quad ;$$

$$\left[(j,\ i):\{(0,\ 7),\ (1,\ 8),\ (2,\ 9),\ (3,\ 10),\ (4,\ 11)\ (5,\ 12)\}\right]$$

Thus by putting the values of $P_0(z,s)$, $P_1(z,s)$, $Q_0(z,s)$, $Q_1(z,s)$ in equations (19) and (20) we have the final approximate solutions for $P(z,s)$ and $Q(z,s)$

$$P(z,s)=\frac{\begin{array}{l}[K_1(z)+\beta K_{13}(z)]\overline{Q}_0(s)+[K_2(z)+\beta K_{14}(z)]\overline{P}_0(s)+[K_3(z)+\beta K_{15}(z)]\overline{P}_M(s)\\+[K_4(z)+\beta K_{16}(z)]\sum_{n=0}^{M}\overline{Q}_n(s)+[K_5(z)+\beta K_{17}(z)]\sum_{n=0}^{M}\overline{P}_n(s)+[K_6(z)+\beta K_{18}(z)]\end{array}}{B(z)}$$

.... (29)

$$Q(z,s)=\frac{\begin{array}{l}[M(z)+\beta K_7(z)]\overline{Q}_0(s)+\beta K_8(z)\overline{P}_0(s)+\beta K_9(z)\overline{P}_M(s)+[N(z)+\beta K_{10}(z)]\sum_{n=0}^{M}\overline{Q}_n(s)\\+\beta K_{11}(z)\sum_{n=0}^{M}\overline{P}_n(s)+\beta K_{12}(z)\end{array}}{A(z)}$$

.... (30)

where

$$P(1,s)=\sum_{n=0}^{M}\overline{P}_n(s)=\frac{s+\alpha+\xi}{s(s+\alpha+\beta+\xi)}$$

$$Q(1,s)=\sum_{n=0}^{M}\overline{Q}_n(s)=\frac{\beta}{s(s+\alpha+\beta+\xi)}$$

On adding relations (29) and (30), we have

$$R(z,s)=\frac{\begin{array}{l}[A(z)\{K_1(z)+\beta K_{13}(z)\}+B(z)\{M(z)+\beta K_7(z)\}]\overline{Q}_0(s)+[A(z)\{K_2(z)+\beta K_{14}(z)\}+\beta B(z)K_8(z)]\overline{P}_0(s)+\\ [A(z)\{K_3(z)+\beta K_{15}(z)\}+\beta B(z)K_9(z)]\overline{P}_M(s)+[A(z)\{K_4(z)+\beta K_{16}(z)\}+B(z)\{N(z)+\beta K_{10}(z)\}]\sum_{n=0}^{M}\overline{Q}_n(s)\\+[A(z)\{K_5(z)+\beta K_{17}(z)\}+\beta B(z)K_{11}(z)]\sum_{n=0}^{M}\overline{P}_n(s)+[A(z)\{K_6(z)+\beta K_{18}(z)\}+\beta B(z)K_{12}(z)]\end{array}}{A(z)B(z)}$$

… (31)

Since,

$$\sum_{n=0}^{M}\overline{R}_n(s)=\sum_{n=0}^{M}\overline{P}_n(s)+\sum_{n=0}^{M}\overline{Q}_n(s)=\frac{1}{s} \quad \text{… (32)}$$

Thus relation (31), for z=1 gives

$$R(1,s) = \frac{1}{s} = \lim_{z\to 1} R(z,s) \qquad \text{... (33)}$$

$$P(0,s) = \overline{P}_0(s) = \lim_{z\to 0} P(z,s) \qquad \text{...... (34)}$$

and $$Q(0,s) = \overline{Q}_0(s) = \lim_{z\to 0} Q(z,s) \qquad \text{..... (35)}$$

The relations (33), (34), and (35) on solution gives the values of $\overline{P}_0(s)$, $\overline{Q}_0(s)$, $\overline{P}_M(s)$..

4. Particular Cases:

Case 1: Letting $\alpha\to\infty$, $\beta\to 0$ and setting $\varepsilon=1, N=1$ and $\mu_1=\mu_2=\mu$ (i.e., when the departure rate is $n\mu$), in relation (31), we have

$$r(z,s) = \frac{L_1(z)\,\overline{P}_M(s) + L_2(z)\frac{1}{s} + L_3(z)}{K(z)} \qquad \text{.... (36)}$$

where

$$L_1(z) = -\frac{\lambda_1}{\mu}\int_0^z z^M \cdot (z-1)^{\frac{s+\xi}{\mu}} \cdot e^{\frac{-\lambda_1}{\mu}\cdot z}\, dz$$

$$L_2(z) = \frac{\xi}{\mu}\int_0^z (z-1)^{\frac{s+\xi}{\mu}-1} \cdot e^{\frac{-\lambda_1}{\mu}\cdot z}\, dz$$

$$L_3(z) = \frac{1}{\mu}\int_0^z (z-1)^{\frac{s+\xi}{\mu}-1} \cdot e^{\frac{-\lambda_1}{\mu}\cdot z}\, dz$$

$$K(z) = (z-1)^{\frac{(s+\xi)}{\mu}} \cdot e^{\frac{-\lambda_1}{\mu}\cdot z}$$

The value of the unknown quantity $\overline{P}_M(s)$ can be obtained by solving the equation $\lim_{z\to 1} r(z,s) = \frac{1}{s}$.

Case 2: In relation (31), if p=1, i.e. when there are no feedback customers. The model reduces to one which is studied by Kumar, Darvinder [9].

5. Steady State Results:

This can at once be obtained by the well -known property of the Laplace transform given below:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,\bar{f}(s), \quad \text{If the limit on the left hand side exists.}$$

Then

$$R(z) = \lim_{s\to 0}\left[s\,R(z,s)\right]$$

By employing this property, we have from relation (31).

$$R(z)=\frac{\begin{array}{l}Q_0\left[A'(z)\{K_1'(z)+\beta K_{13}'(z)\}+B'(z)\{M'(z)+\beta K_7'(z)\}\right]+P_0\left[A'(z)\{K_2'(z)+\beta K_{14}'(z)\}+\beta B'(z)K_8'(z)\right]\\+P_M\left[A'(z)\{K_3'(z)+\beta K_{15}'(z)\}+\beta B'(z)K_9'(z)\right]+\sum_{n=0}^{M}Q_n\left[A'(z)\{K_4'(z)+\beta K_{16}'(z)\}+B'(z)\{N'(z)+\beta K_{10}'(z)\}\right]\\+\sum_{n=0}^{M}P_n\left[A'(z)\{K_5'(z)+\beta K_{17}'(z)\}+\beta B'(z)K_{11}'(z)\right]+C\end{array}}{A'(z)\ B'(z)}$$

.....

(37)

where,

$$A'(z)=z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(\alpha+\xi)}{\mu_2\varepsilon}}$$

$$B'(z)=z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{\xi}{\mu_2\varepsilon}}\cdot\ e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}$$

$$M'(z)=\frac{1-\varepsilon N}{\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}-1}\cdot(z-1)^{\frac{(\alpha+\xi)}{\mu_2\varepsilon}}\cdot dz$$

$$N'(z)=\frac{\xi}{\mu_2\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(\alpha+\xi)}{\mu_2\varepsilon}-1}\ dz$$

$$K_1'(z)=\frac{\alpha}{\mu_1 p\varepsilon}\int_0^z (z-1)^{\frac{\xi}{\mu_1 p\varepsilon}-\left\{\frac{(\alpha+\xi)}{\mu_2\varepsilon}+1\right\}}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}\cdot M'(z)\ dz$$

$$K_2'(z)=\frac{1-\varepsilon N}{\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}-1}\cdot(z-1)^{\frac{\xi}{\mu_1 p\varepsilon}}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}\ dz$$

$$K_3'(z)=-\frac{\lambda_1}{\mu_1 p\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}+M}\cdot(z-1)^{\frac{\xi}{\mu_1 p\varepsilon}}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}\ dz$$

$$K_4'(z)=\frac{\alpha}{\mu_1 p\varepsilon}\int_0^z (z-1)^{\frac{\xi}{\mu_1 p\varepsilon}-\left\{\frac{(\alpha+\xi)}{\mu_2\varepsilon}+1\right\}}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}\cdot N'(z)\ dz$$

$$K_5'(z)=\frac{\xi}{\mu_1 p\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{\xi}{\mu_1 p\varepsilon}-1}\cdot e^{\frac{-\lambda_1}{\mu_1 p\varepsilon}\cdot z}\ dz$$

$$K_{i+6}'(z)=\frac{1}{\mu_2\varepsilon}\int_0^z z^{\frac{1-\varepsilon N}{\varepsilon}}\cdot(z-1)^{\frac{(\alpha+\xi)}{\mu_2\varepsilon}-1}\cdot K_i'(z)\ dz\qquad;\ i=1,2,3,4,5.$$

$$K_{i+j+(6-j)}'(z)=\frac{1}{\mu_1 p\varepsilon}\int_0^z\frac{1}{z-1}\left[\frac{\alpha B'(z)}{A'(z)}K_i'(z)-K_{j+1}'(z)\right]dz\quad;$$

$$\left[(j,i):\{(0,7),(1,8),(2,9),(3,10),(4,11)\}\right]$$

C = the constant of integration.

The unknown quantities P_0, Q_0, P_M, $\sum_{n=0}^{M}Q_n$ and $\sum_{n=0}^{M}P_n$ can be evaluated as before.

Particular Case:

Relation (36), on applying the theory of Laplace transforms gives

$$r(z) = \frac{L_1'(z)P_M + L_2'(z) + C'}{K'(z)}$$ ..... (38)

Where

$$r(z) = \lim_{s \to 0} s \cdot r(z,s)$$

$$L_1'(z) = -\frac{\lambda_1}{\mu}\int_0^z z^M \cdot (z-1)^{\frac{\xi}{\mu}} \cdot e^{\frac{-\lambda_1}{\mu} \cdot z}\, dz$$

$$L_2'(z) = \frac{\xi}{\mu}\int_0^z (z-1)^{\frac{\xi}{\mu}-1} \cdot e^{\frac{-\lambda_1}{\mu} \cdot z}\, dz$$

$$K'(z) = (z-1)^{\frac{\xi}{\mu}} \cdot e^{\frac{-\lambda_1}{\mu} \cdot z}$$

C' = the constant of integration.

The unknown quantity of equation (38) can be evaluated as before.

When no catastrophe and feedback are allowed in the queueing system i.e., $\xi = 0$ **and p=1,** relation (38), gives

$$e^{\frac{-\lambda_1}{\mu} z} r(z) = C_1 - \frac{\lambda_1}{\mu} P_M \int z^M e^{\frac{-\lambda_1 z}{\mu}}\, dz$$ .... (39)

where

C_1 = the constant of integration.

The unknown quantity P_M can be evaluated as before.

For unlimited waiting space, the relation (39) becomes, If Max (ρ, |z|) < 1.

$$e^{\frac{-\lambda_1 z}{\mu}} r(z) = C_1$$

Which for z = 1 gives, $C_1 = e^{\frac{-\lambda_1}{\mu}}$

Hence, $r(z) = e^{\frac{-\lambda_1}{\mu}(1-z)}$ ….. (40)

which is a well- known result.

Steady-state probabilities of the M/M/1 Queue:

In [11], Kumar, B. K., and Arivudainambi, D. have studied the transient solution of an M/M/1 queue with catastrophes. They have also obtained the steady- state probabilities and mean & variance of the M/M/1 queue with catastrophes. When a catastrophe occurs at the service facility i.e. $\xi > 0$, the steady- state distribution $\{p_n;\ n \geq 0\}$ of the M/M/1 queue with catastrophes corresponds to

$$p_0 = (1-\rho)\ ;\ n = 0$$ ….(41)

$$p_n = (1-\rho)\rho^n\ ;\ n = 1, 2, 3, \ldots\ldots$$ ….(42)

where

$$\rho = \frac{(\lambda+\mu+\xi)-\sqrt{\lambda^2+\mu^2+\xi^2+2\lambda\xi+2\mu\xi-2\lambda\mu}}{2\mu} \quad \ldots(43)$$

Thus equations (41)-(43) provide the steady- state distribution for the queueing system. Obviously, the steady state distribution exists if and only if $\rho < 1$.

Note: The steady- state probability of this Markov process exists if and only if $\xi > 0$ or $\xi = 0$ and $\lambda > \mu$. It is also observed that the results of equations (41)-(43) agree with the model discussed above and with Chao, X [4].

6. Application of the Model:

The model's real-world scenario may be presented in many practical situations e.g. for a production firm engaged in manufacturing the product 'X'. If a customer finds the service terminal/ sales department, they will be welcomed by a sales manager, who will provide the customer with the first service, such as sale of product 'X', and then the sales manager will provide the second service, i.e., the packing/billing of product 'X'. After receiving first service from the sales manager, the customer may leave the system or join for the second service. After obtaining service, if the customers are unsatisfied (feedback), they will return back to the service terminal as a new arrival or they will leave the service station permanently as satisfied customers. The production firm may for various seasons stop the supply of the product at some time which make them zero instantaneously and start it again can be regarded as the occurrence of a catastrophe. Therefore, the proposed model is highly useful for dealing with real-world queueing situations in the environmental changing states and possibilities of catastrophes and feedback such as in manufacturing firms, banking or health sectors etc.

7. Conclusion:

In the present paper, we have established a finite capacity queueing system with catastrophes, feedback, state dependent service and environmental change and obtained the transient state solution. Also, steady state results and some interesting particular cases with (without) catastrophes and feedback are derived and discussed.

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