Original

Inverse Domination on a Middle graph of some cycle related graphs

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Abstract
Let G be non-trivial graph. A dominating set for a graph G is a subset D of its vertices, such that any vertex of G is in D, or has a neighbor in D. The set S⊂ V(G)- D such that S is a dominating set of G, then S is called an inverse dominating set with respect to D. The minimum cardinality of an inverse dominating set is called an inverse dominating number and is denoted by γ^' (G). In this paper we find inverse domination number on Middle graph of some graphs.

Keyword: *Domination, Inverse Domination, Middle Graph.*

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1. INTRODUCTION

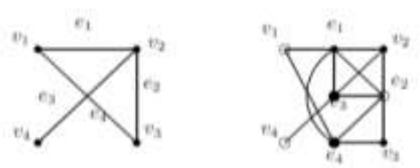
Domination is a graph theoretic concept introduced by C. Berge in 1958 and O. Ore in 1962. It was O. Ore [4] who introduced the term dominating set and domination number. The paper of Kulli and Singarkanti [3] in 1991 which initiated the study of inverse domination in graphs. Let D be a minimum dominating set of G. If $V - D$ contains a dominating set S, then S is called an inverse dominating sets of G with respect to D. The inverse dominating number $\gamma^{-1}(G)$ is the minimum cardinality taken over all the minimal inverse dominating sets of G. The middle graph $M(G)$ of a graph G has been introduced by **T. Hamada and I. Yoshimura** in [2]. S. B. Chikkodimath and E. Sampathkumar also studied it independently, and they called it the semitotal graph $T_1(G)$ of a graph G in [1]. We define a graph $M(G)$ as the vertex set $V(M(G)) = \{x, y/x \in V(G), y \in E(G)\}$ and the edge set $E(M(G)) = \{xy\}$ where $x, y \in E(G)$ or $x \in V(G), y \in E(G)$. Then $M(G)$ is called the middle graph of G.

1.1 Definition and Examples

Definition 1.1. Let G be a graph with vertex set $V(G)$ and the edge set $E(G)$. The middle graph of the graph G denoted by $M(G)$ is defined as follows: The vertex set of $M(G)$ is $V(G) \cup E(G)$ in which two vertices x and y are adjacent in $M(G)$ if the following conditions hold

1. $x\,y \in E(G)$.
2. $x \in V(G),\ y \in E(G)$ and they are incident in G.

Example 1.2.
Consider the graphs G and $M(G)$ given in figure clearly $\{e_3, e_4\}$ is a minimum dominating set of $M(G)$ and $\{v_1, v_4, e_2\}$ is a corresponding minimum inverse dominating set. Here $\gamma'(M(G) = 2$.



G $M(G)$

2. Inverse Domination on Middle Graphs

Theorem: 2.1

For a cycle C_n, $\gamma'(M(C_n)) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, where $n \geq 2$.

Proof: Let the cycle C_n be $v_1v_2\ldots v_nv_1$ and $M(C_n)$ be the middle graph of C_n. Clearly $V(M(C_n)) = \{v_i, e_i / 1 \leq i \leq n\}$ and $E\big(M(C_n)\big) = \{e_ie_{i+1}, v_ie_i, v_{i+1}e_i \,/\, 1 \leq i \leq n\}$ and $e_i = v_iv_{i+1}$, where the suffixes modulo n. Let D be a minimum isolate dominating set of $M(P_n)$ and S be a corresponding minimum inverse dominating set with respect to D. We now consider the following two cases.

Case 1. n is even

In this case a minimum dominating set of $M(C_n)$ is $D = \{e_1, e_3, e_5, \ldots, e_{n-3}, e_{n-1}\}$ and a corresponding minimum inverse dominating set $S = \{e_2, e_4, e_6, \ldots, e_{n-2}, e_n\}$. Since $|S| = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Hence $\gamma'\big(M(C_n)\big) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Case 2. n is odd

In this case a minimum dominating set of $M(C_n)$ is $D = \{v_n, e_1, e_3, e_5, \ldots, e_{n-4}, e_{n-2}\}$ and a corresponding minimum inverse dominating set $S = \{v_1, e_2, e_4, e_6, \ldots, e_{n-3}, e_{n-1}\}$. Since $|S| = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Hence $\gamma'(M(C_n)) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Theorem: 2.2

For the wheel graph W_n, $\gamma'(M(W_n)) = \left\lceil \frac{n}{2} \right\rceil$.

Proof: Let $v_1v_2 \ldots v_{n-1}v_1$ be the cycle C_{n-1}. Add a vertex v which is adjacent to $v_i, 1 \leq i \leq n-1$. The resultant graph is the wheel graph W_n with vertex set $V(W_n) = \{v, v_i / 1 \leq i \leq n-1\}$ and edge set $E(W_n) = \{e_{0,i}, e_i / 1 \leq i \leq n-1\}$ where $e_i = v_iv_{i+1}$ and $e_{0,i} = vv_i$, the suffixes modulo $n-1$.

Let $M(W_n)$ be the middle graph of W_n. Clearly $V\big(M(W_n)\big) = \{v, v_i, e_i, e_{0,i}, 1 \leq i \leq n-1\}$ and $E(M(W_n)) = \{e_ie_{i+1}, e_{0,i}e_{0,j}, e_ie_{0,i}, e_ie_{0,i+1}, v_ie_i, v_ie_{i-1}, v_ie_{0,i}, ve_{0,i} \,/\, 1 \leq i,j \leq n-1, i \neq j\}$, where the suffixes modulo n. Let D be a minimum isolate dominating set of $M(W_n)$ and S be a corresponding minimum inverse dominating set with respect to D. We now consider the following two cases depends on n.

Case 1. n is even

In this case a minimum isolate dominating set D of $M(W_n)$ is $\{e_2, e_4, e_6, \ldots, e_{n-1}, e_{0,1}\}$ and a corresponding minimum inverse dominating set $S = \{e_1, e_3, e_5, \ldots, e_{n-2}, e_{0,n-1}\}$. Clearly $|S| = \left\lceil \frac{n}{2} \right\rceil$. Therefore, $\gamma'(M(W_n)) = \left\lceil \frac{n}{2} \right\rceil$.

Case 2. n is odd

Here a minimum isolate dominating set $D = \{e_2, e_4, e_6, \ldots, e_{n-2}, e_{0,1}\}$ and a minimum inverse dominating set S' with respect to D is $S = \{e_1, e_3, e_5, \ldots, e_{n-3}, e_{0,n-1}\}$. Clearly $|S| = \left\lceil \frac{n}{2} \right\rceil$. Therefore, $\gamma'(M(W_n)) = \left\lceil \frac{n}{2} \right\rceil$.

The theorem follows from the cases 1 and 2.

Theorem: 2.3
Let W_n be a wheel graph of order n and G be a graph obtained by removing a vertex u from W_n. Then $\gamma'(M(G)) = \left\lfloor \frac{n}{2} \right\rfloor$.
Proof: Let $v_1 v_2 \ldots v_{n-1} v_1$ be the cycle C_{n-1}. Add a vertex v which is adjacent to $v_i, 1 \le i \le n-1$. The resultant graph is the wheel graph W_n with vertex set $V(W_n) = \{v, v_i / 1 \le i \le n-1\}$ and edge set $E(W_n) = \{e_{0,i}, e_i / 1 \le i \le n-1\}$ where $e_i = v_i v_{i+1}$ and $e_{0,i} = vv_i$, the suffixes modulo $n-1$. Now G be a graph obtained by removing a vertex u from W_n. We now consider the following two cases.
Case 1. $u = v$
In this case the resulting graph $G = C_{n-1}$. By theorem 2.2, $\gamma'(M(G)) = \left\lfloor \frac{n}{2} \right\rfloor$.
Case 2. $u = v_i, 1 \le i \le n-1$
Here we fix $u = v_1$. Now the vertex set $V(G) = \{v_i / 2 \le i \le n-1\}$ and $E(G) = \{e_{0,i}, e_i / 2 \le i \le n-1\}$ where $e_i = v_i v_{i+1}, e_{0,i} = vv_i$ and the suffixes modulo $n-1$. Let $W(G)$ be the middle graph of G. Clearly $V(M(G)) = \{v, v_i, e_i, e_{0,i} / 2 \le i \le n-1\}$ and $E(M(G)) = \{e_i e_{i+1}, e_{0,i} e_{0,j}, e_i e_{0,i}, e_i e_{0,i+1}, v_i e_i, v_i e_{i-1}, v_i e_{0,i}, v e_{0,i} / 1 \le i, j \le n-1, i \ne j\}$,
where the suffixes modulo n. Let D be a minimum isolate dominating set of $M(G)$ and S be a corresponding minimum inverse dominating set with respect to D. We now consider the following two subcases.
Sub Case 2.1 $G = W_4 - v_1$
In this case the minimum dominating set of $M(G)$, $D = \{e_2, v\}$ and the corresponding minimum inverse dominating set of $M(G)$ is either $\{e_{0,2}, e_{0,3}\}$ or $\{e_0, v_3\}$. Clearly $|S| = 2$. Hence $\gamma'(G) = \gamma'(W_4 - v_1) = 2$.
Sub Case 2.2 $G = W_n - v_1,\ n \ge 5$
In this case, if n is odd the minimum dominating set D of $M(G)$ is $\{e_2, e_4, e_6, \ldots, e_{n-3}, e_{0,n-1}\}$ and the corresponding minimum inverse dominating set $S = \{e_3, e_5, e_7, \ldots, e_{n-2}, e_{0,2}\}$. If n is even the minimum dominating set D of $M(G)$ is $\{e_2, e_4, e_6, \ldots, e_{n-2}, e_{0,n-1}\}$ and the corresponding minimum inverse dominating set $S = \{e_3, e_5, e_7, \ldots, e_{n-3}, e_{0,2}, v_{n-1}\}$.
From the above cases $|S| = \left\lfloor \frac{n}{2} \right\rfloor$. Hence $\gamma'(G) = \left\lfloor \frac{n}{2} \right\rfloor$.
Thus the theorem follows from case 1 and 2.

Theorem: 2.4
For the jewel graph $J_n, \gamma'(M(J_n)) = n + 2$.
Proof: Consider the 4- cycle $xwyux$. Join x and y. Add new vertices $v_i, 1 \le i \le n$ and join v_i to both u and w. The resulting graph is the jewel graph J_n with vertex set $V(J_n) = \{x, y, u, w, v_i / 1 \le i \le n\}$ and edge set $E(J_n) = \{xy, xu, xw, yv, yw, uv_i, wv_i / 1 \le i \le n\}$. Denote xw by e_1, wy by e_2, uy by e_3, xu by e_4, xy by e_{xy}, uv_i by e_i' and wv_i by $e_i'', 1 \le i \le n$. Let $M(J_n)$ be the middle graph of J_n. Clearly $V(M(J_n)) = \{x, y, u, w, v_i, e_{xy}, e_j, e_i', e_i'' \backslash 1 \le j \le 4, 1 \le i \le n\}$ and $E(M(J_n)) = \{xe_1, xe_4, xe_{xy}, ye_2, y\,e_3, ye_{xy}, ue_3, ue_4, ue_i', we_1, we_2, we_i'', v_i e_i', v_i e_i'' / 1 \le i \le n\}$. In $M(J_n)$, $\langle \{e_1', e_2', \ldots, e_n'\} \rangle = K_n$ and $\langle \{e_1'', e_2'', \ldots, e_n''\} \rangle = K_n$. Clearly $D = \{e_1, e_3, e_1', e_2', \ldots, e_n'\}$ is a minimum isolate dominating set and $S = \{e_2, e_4, e_1'', e_2'', \ldots, e_n''\}$ is

a minimum inverse dominating set S with respect to D. Since $|S| = n + 2$. Hence $\gamma'\big(M(J_n)\big) = n + 2$.

Theorem: 2.5

For the pan graph $P_{n,1}$, $\gamma'(M(P_{n,1})) = \left\lceil\frac{n}{2}\right\rceil + 1$.

Proof: Consider the cycle $u_1u_2\ldots u_nu_1$. Join a new vertex u_0 to the vertex u_1. The resultant graph is the pan graph $P_{n,1}$ with a vertex set $V(P_{n,1}) = \{u_i/0 \le i \le n\}$ and the edge set $E(P_{n,1}) = \{u_0u_1, u_iu_{i+1}, u_1u_n/1 \le i \le n-1\}$. Clearly, $V(M(P_{n,1})) = \{u_i, e_i/0 \le i \le n\}$ and the edge set $E(M(P_{n,1})) = \{e_0e_k, e_ie_{i+1}, u_ke_0, u_ie_i, u_ie_{i-1}/1 \le i \le n, k = 1, n\}$, where $e_i = u_iu_{i+1}, e_0 = u_0u_i$ and the suffixes modulo n. Let D be a minimum isolate dominating set of $M\big(P_{n,1}\big)$ and S be a corresponding minimum inverse dominating set with respect to D. We now consider the following two cases.

Case 1. n is odd

In this case a minimum isolate dominating set D of $M(P_{n,1})$ is $\{e_0, e_2, e_4, \ldots, e_{n-3}, e_{n-1}\}$ and a corresponding minimum inverse dominating set $S = \{u_0, e_1, e_3, e_5, \ldots, e_{n-2}, e_n\}$. Clearly $|S| = \left\lceil\frac{n}{2}\right\rceil + 1$. Hence $\gamma'(M(P_{n,1})) = \left\lceil\frac{n}{2}\right\rceil + 1$.

Case 2. n is even

In this case a minimum isolate dominating set D of $M\big(P_{n,1}\big)$ is $\{e_0, e_2, e_4, \ldots, e_{n-2}, e_n\}$ and a corresponding minimum inverse dominating set $S = \{u_0, e_1, e_3, e_5, \ldots, e_{n-3}, e_{n-1}\}$. Hence $\gamma'(M(P_{n,1})) = |S| = \left\lceil\frac{n}{2}\right\rceil + 1$.

Theorem: 2.6

Let G be a Peterson graph of order 10. Then $\gamma'(M(G)) = 6$.

Proof: Let G be a Peterson graph of order 10. Clearly $V(G) = \backslash\{u_i, v_i/1 \le i \le 5\}$ and $E(G) = \{u_iv_i, v_iv_{i+2}, v_iv_{i+3}, u_iu_{i+1} / 1 \le i \le 5\}$, where the suffixes modulo 5.

Denote $u_iv_i = e''_{i,i}, v_iv_{i+2} = e'_{i,i+2}, v_iv_{i+3} = e'_{i,i+3}, u_iu_{i+1} = e_{i,i+1}$. Let $M(G)$ be the middle graph of G.

Clearly $V(M(G) = \{u_iv_i, e''_{i,i}, e'_{i,i+2}, e'_{i,i+3}, e_{i,i+1}/1 \le i \le 5\}$ and $E\big(M(G)\big) = \{u_ie_{i,i+1}, u_ie''_{i,i}, v_ie''_{i,i}, v_ie'_{i,i+2}, v_ie'_{i,i+3}, e_{i,i+1}e_{i+1,i+2}, e_{i,i+1}e''_{i,i}, e_{i,i+1}e''_{i+1,i+1}, e''_{i,i}e'_{i,i+2}, e''_{i,i}e'_{i,i+3}/ 1 \le i \le 5\}$ where the suffixes modulo 5. Let D be the minimum dominating set of $M(G)$ and S be the minimum inverse dominating set of $M(G)$. Now $D = \{u_2, u_4, e'_{1,4}, e''_{1,1}, e''_{3,3}\}$ and $S = \{u_1, e_{4,4}, e'_{1,3}, e'_{2,5}, e''_{2,2}, e''_{4,4}\}$. Clearly $|S| = 6$. Hence $\gamma'(M(G)) = 6$.

2. REFERENCES

[1] Chikkodimath, S, B and Sampathkumar, E, "Semitotal Graphs-Il, Graph theory Research Report", Kar Datak University No. 2, pp.5-9, 1973.

[2] Hamada, T and Yoshimura, I, Traversabiliy and connectiviy of the middle graphs of a graph, Discrete Mathematics, Vol.14 (196), pp. 247-255, 1976.

[3] V. R. Kulli, S. C. Sigarkant, Inverse domination in graphs, Nat. Acad Sci. Letters, Vol. 14, 473-475, 1991.

[4] O. Ore, Theory of Graphs, American Mathematical Society, Provedence, R. I., 1962.