PICTURE FUZZY ACZEL-ALSINA POWER GEOMETRIC AGGREGA- TION OPERATORS AND THEIR APPLICATION TO MULTI CRITERIA DECISION-MAKING

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Abstract. This paper develops a robust picture fuzzy (PF) decision-making framework by integrating power aggregation operators derived from Acz´el-Alsina operations. The proposed power aggregation operators effec- tively capture intricate interrelationships among multiple criteria, thereby enhancing the precision and reliability of decision-making processes. In this study, the familiarity of decision makers with the evaluated objects is systematically incorporated into the PF framework alongside primary data, ensuring a more comprehensive as- sessment. Motivated by the operational principles of Acz´el-Alsina functions, this research advances the theoretical foundation of PF Acz´el-Alsina power-weighted and ordered-weighted geometric operators, seamlessly integrating decision makers' expertise into the aggregation process. The structural properties and mathematical character- istics of these newly developed operators are rigorously analyzed. To validate their practical applicability, we employ the proposed operators to solve a complex multi-criteria decision-making (MCDM) problem within the food industry, a domain where uncertainty and nuanced judgments play a critical role. A comparative evaluation against existing operators highlights the superior performance of our approach in effectively managing uncertainty, refining decision accuracy, and enhancing adaptability to real-world decision-making challenges.

Keywords: Multi criteria decision-making, picture fuzzy set, Aczel-Alsina t-norms and t-conorms, Power aggre- gation operator, Food industry.

1 Introduction

Multi-criteria decision-making (MCDM) is a fundamental aspect of decision theory with extensive applications across diverse domains, including engineering, economics, healthcare, and environmental management. Decision- makers frequently encounter challenges in deriving logical conclusions due to the complexity of evaluating multiple conflicting criteria and assessing numerous alternatives under uncertainty. Traditional decision-making models often struggle to accommodate the vagueness and subjectivity inherent in human judgment, necessitating the development of more flexible mathematical frameworks.

To address these challenges, Zadeh (1965) introduced fuzzy set (FS) theory, which allows for a more nuanced representation of uncertainty by incorporating the degree of membership μ within the range $0 \leq \mu \leq 1$. This approach enabled the modeling of imprecise information, making it a powerful tool in decision analysis. However, FS theory does not account for the possibility of hesitation or partial membership in multiple categories.

To enhance the expressive power of FS theory, Atanassov (1986) proposed intuitionistic fuzzy sets (IFS), which introduce the degree of non-membership ν alongside the membership degree μ, subject to the constraint $0 \leq \mu + \nu \leq 1$. IFS allows decision-makers to express hesitation by acknowledging both acceptance and rejection degrees, improving the representation of uncertain scenarios. Nevertheless, IFS still lacks the ability to explicitly incorporate neutrality, which is essential in many real-world decision-making problems where experts may prefer an intermediate stance rather than a strict ”yes” or ”no” response.

To overcome this limitation, Cuong et al. (2013) introduced picture fuzzy sets (PFS), an extension of IFS that incorporates an additional parameter: the degree of neutrality (η), alongside membership (μ) and non-membership (ν), under the condition $0 \leq \mu + \eta + \nu \leq 1$. PFS models are particularly effective in scenarios where human opinions involve nuanced responses such as ”yes,” ”no,” ”neutral,” or even cases where decision-makers partially agree and partially disagree simultaneously. These sets have proven useful in areas such as expert systems, medical diagnosis, risk assessment, and social decision-making, where neutrality and hesitation play a crucial role.

Additionally, some individuals may choose not to express a preference (π). Cuong et al. (2016) examined the classification and properties of representable picture t-norms and picture t-conorms. Yager (1988) introduced the 1
ordered weighted aggregation (OWA) operator and analyzed its characteristics. Further, Wei (2017) investigated PF arithmetic and geometric operators, including PF hybrid aggregation operators (AOs) and their applications. Several researchers have contributed to the advancement of PF aggregation operators in MCDM. Garg (2017) proposed a range of AOs for PFSs and applied them to solve MCDM problems. Wei (2018) explored MCDM using arithmetic and geometric AOs based on Hamacher operations. Khan et al. (2019) introduced PF Einstein operations, while Zhang et al. (2018) proposed PF Dombi Heronian mean operators. Jana et al. (2019) developed PF Hamacher AOs for enterprise performance evaluation. Other notable contributions include works by Ates et al. (2020), Wang et al. (2018), and Qiyas et al. (2020), who introduced various PF aggregation operators for handling complex decision-making scenarios. This paper introduces a novel class of aggregation operators by integrating the power aggregation (PA) operator with Acz´el-Alsina power (AAP) operations, thereby enhancing information aggregation in the PF environment. The proposed PF AOs include the PF Acz´el-Alsina power weighted geometric (PFAAPWG) operator and the PF Acz´el-Alsina power ordered weighted geometric (PFAAPOWG) operator. Additionally, the study explores their fundamental properties and practical applications in MCDM problems. The organization of the paper are showed in Fig. 1

1.1 Motivation

The motivation behind this research stems from the need to enhance decision-making methodologies in complex and uncertain environments. The key motivating factors are as follows:

- Expanding the PF framework by incorporating advanced aggregation mechanisms is crucial for handling uncertainty and imprecise information in real-world applications.
- A significant research gap exists in the integration of Acz´el-Alsina power

aggregation operators within PF environments, necessitating a systematic exploration of their applicability.

The verification of the mathematical properties of the proposed operators is essential to ensure their robustness and reliability in decision-making contexts.

This study aims to develop novel PF Acz´el-Alsina power geometric operators, extending their theoretical foundation and practical relevance to diverse decision-making scenarios.

A comparative evaluation is conducted to highlight the advantages of the proposed operators over existing approaches.

1.2 Contribution

The major contributions of this paper are summarized as follows:

The introduction of PF Acz´el-Alsina power-weighted and ordered-weighted geometric aggregation opera- tors, providing a new approach to information fusion in PF environments.

A comprehensive theoretical analysis of the fundamental properties of the proposed operators, ensuring their mathematical soundness and applicability.

The development of a novel MCDM approach utilizing PF Acz´el-Alsina power geometric aggregation operators to enhance decision-making precision.

The application of the proposed methodology to a real-world decision-making problem in the food industry, demonstrating its effectiveness through comparative analysis against existing aggregation techniques.

1.3 Organization

This paper is systematically structured into seven sections:

Section 2 provides an overview of the fundamental concepts of PFS and their associated operations, forming the theoretical foundation of this study.

Section 3 introduces the proposed PF Acz´el-Alsina power geometric aggregation operators and discusses their formulation.

Section 4 presents a structured algorithm for solving MCDM problems using the proposed aggregation framework.

Section 5 demonstrates a numerical case study in the PF context.

Section 6 Comparative analysis showcasing the advantages of the proposed approach.

Section 7 concludes the study with a summary of key findings and potential directions for future research.

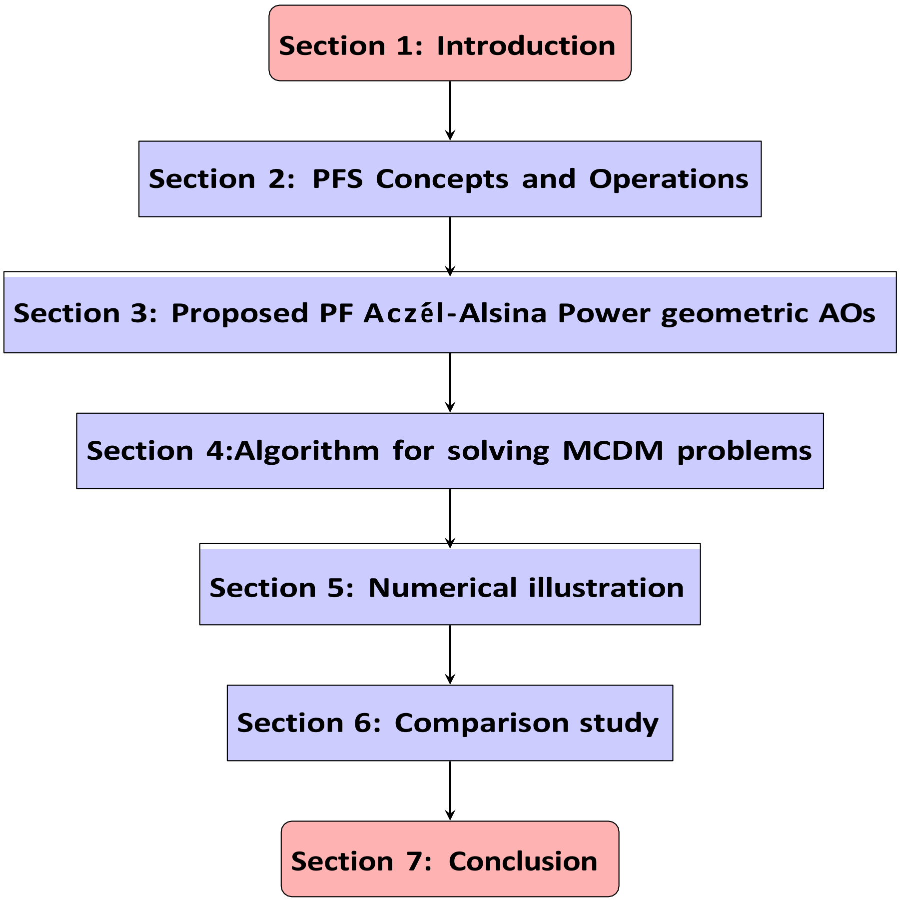

Figure 1: Flowchart illustrating the organization of the paper.

2 Preliminaries

In this section, some basic concepts have been reviewed related to PFS.

2.1 Picture Fuzzy Set

The PFS (Cuong et al. (2013, 2014)) is an extension of IFS. The mathematical form of PFS is expressed as follows:

Definition 1. *A picture fuzzy set A on universal set X is defined by,*

$$A = \{\langle x, \mu A(x), \eta A(x), \nu A(x)\rangle / x \in X\}$$

Where, $\mu A(x)$, $\eta A(x)$ *and* $\nu A(x) \in [0, 1]$ *are the degree of membership, the degree of neutral membership and the degree of non-membership of* $x \in A$ *respectively, with the following condition:* $0 \leq \mu A(x) + \eta A(x) + \nu A(x) \leq 1\ \forall x \in X$. *Then, for* $x \in X$, $\pi A(x) = 1 - \mu A(x) - \eta A(x) - \nu A(x)$ *could be called the degree of refusal membership of x in* A. *For convenience,* $\alpha = (\mu\alpha, \eta\alpha, \nu\alpha)$ *is called a picture fuzzy number (PFN).*

2.2 Comparison for PFNs

According to Garg (2017) the score and accuracy functions of PFNs are as follows:

Definition 2. *Let* $\alpha = (\mu\alpha, \eta\alpha, \nu\alpha)$ *be a PFN, its score function* $S(\alpha)$ *and its accuracy function* $A(\alpha)$ *is defined by* $S(\alpha) = \mu\alpha - \eta\alpha - \nu\alpha$; $S(\alpha) \in [-1, 1]$, $A(\alpha) = \mu\alpha + \eta\alpha + \nu\alpha$; $A(\alpha) \in [0, 1]$.

Based on the $S(\alpha)$ and $A(\alpha)$ an order relationship between two PFNs is defined as follows.

Definition 3. *Let $\alpha_1 = (\mu_{\alpha 1}, \eta_{\alpha 1}, v_{\alpha 1})$ and $\alpha_2 = (\mu_{\alpha 2}, \eta_{\alpha 2}, v_{\alpha 2})$ be two PFNs. Then the following comparison rules can be used:*

- *(i) If $S(\alpha 1) < S(\alpha 2)$ then $\alpha 1 \prec \alpha 2$*
- *(ii) f $S(\alpha 1) = S(\alpha 2)$ then*
  - *(a) If $A(\alpha 1) < A(\alpha 2)$ then $\alpha 1 \prec \alpha 2$*
  - *(b) If $A(\alpha 1) = A(\alpha 2)$ then $\alpha 1 \approx \alpha 2$.*

2.3 Operation laws of Picture fuzzy numbers

Naeem (2022) operations laws and power geometric aggregation operator (Ullah, K. et all (2023)) are defined for PFNs as follows.

Definition 4. *The PG operator are defined as:*

$$PG(\breve b_1, \breve b_2, \ldots, \breve b_r) = \prod_{k=1}^{r} \breve b_k^{\frac{r(1+\ddot U(\breve b_k))}{\sum_{k=1}^{r} (1+\ddot U(\breve b_k))}} \tag{1}$$

where $\ddot U(\breve b_k) = \sum_{k=1, k\neq h}^{r} sup(\breve a_h, \breve a_k)$, $sup(\breve a_h, \breve a_k) = 1 - D(\breve a_h, \breve a_k)$ *and the weight* $\frac{1+\ddot U(\breve b_h)}{\sum_{k=1}^{r} 1+\ddot U(\breve b_k)}$ *of the argument* $\breve b_h$ *depends on all the input arguments* $\breve b_k (k = 1, 2, \ldots, r)$, *which enables the argument values to support each other in the geometric aggregation process.*

Definition 5. *Let $\alpha_1 = \langle \mu_{\alpha 1}, \eta_{\alpha 1}, v_{\alpha 1} \rangle$ and $\alpha_2 = \langle \mu_{\alpha 2}, \eta_{\alpha 2}, v_{\alpha 2} \rangle$ be any two PFNs, then the Euclidean distance between them is defined as follows:*

$$D(\alpha_1, \alpha_2) = \frac{1}{3}\{|\mu_{\alpha 1} - \mu_{\alpha 2}| + |\eta_{\alpha 1} - \eta_{\alpha 2}| + |v_{\alpha 1} - v_{\alpha 2}|\} \tag{2}$$

Definition 6. *AA in early 1982 introduced the concepts of TN and TCN classes for functional equations. The AATN can be defined as follows:*

$$\mathrm{Eg}^{\chi}(\ddot{\breve u}, \ddot{\breve v}) = \begin{cases} \mathrm{Eg}_D(\ddot{\breve u}, \ddot{\breve v}), & \text{if } \beta = 0 \\ min(\ddot{\breve u}, \ddot{\breve v}), & \text{if } \beta = \infty \\ e^{-\{(-Ln\ddot u)^{\beta} + (-Ln \ddot v)^{\beta}\}^{\frac{1}{\beta}}}, & \text{otherwise.} \end{cases}$$

and the AATCN can be defined as follows:

$$\mathrm{O}^{\chi}(\ddot{\breve u}, \ddot{\breve v}) = \begin{cases} \mathrm{O}_D(\ddot{\breve u}, \ddot{\breve v}); & \text{if } \beta = 0 \\ max(\ddot{\breve u}, \ddot{\breve v}); & \text{if } \beta = \infty \\ e^{-\{(-Ln(1-\ddot{\breve u}))^{\beta} + (-Ln(1-\ddot{\breve v}))^{\beta}\}^{\frac{1}{\beta}}}, & \text{otherwise.} \end{cases}$$

such that $\mathrm{Eg}^0 = \mathrm{Eg}$, $\mathrm{Eg}^1 = \mathrm{Eg}$, $\mathrm{Eg}^{\infty} = min$, $\mathrm{O}^0 = \mathrm{O}_D$, $\mathrm{O}^1 = \mathrm{O}_P$, $\mathrm{O}^{\infty} = max$. *The*

TN Eg^{β} *and are* *TCN* O^{β}

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combained to one another for each β ∈ [0, ∞]. The class of AATN is strictly increasing, and the class of AATCN is strictly decreasing. The following is the AATN and AATCN operational laws in connection with PF theory.

Definition 7. *Let* $\alpha j = \langle \mu\alpha j , \nu\alpha j , \eta\alpha j \rangle$, $j = 1, 2$ *be two PFNs,* $\beta \geq 1$ *and* $K > 0$. *Then, the AATN and AATCN operations of PFN are defined as:*

1. $\alpha 1 \oplus \alpha 2 = \langle 1-e^{-\{(-Ln(1-\mu\alpha 1))^{\beta}+(-Ln(1-\mu\alpha 2))^{\beta}\}^{\frac{1}{\beta}}}, e^{-\{(-Ln(\eta\alpha 1))^{\beta}+(-Ln(\eta\alpha 2))^{\beta}\}^{\frac{1}{\beta}}}, e^{-\{(-Ln(\nu\alpha 1))^{\beta}+(-Ln(\nu\alpha 2)))^{\beta}\}^{\frac{1}{\beta}}} \rangle;$
2. $\alpha 1 \otimes \alpha 2 = \langle e^{-\{(-Ln(\mu\alpha 1))^{\beta}+(-Ln(\mu\alpha 2))^{\beta}\}^{\frac{1}{\beta}}}, 1-e^{-\{(-Ln(1-\eta\alpha 1))^{\beta}+(-Ln(1-\eta\alpha 2))^{\beta}\}^{\frac{1}{\beta}}}, 1-e^{-\{(-Ln(1-\nu\alpha 1))^{\beta}+(-Ln(1-\nu\alpha 2)))^{\beta}\}^{\frac{1}{\beta}}} \rangle;$
3. $K \cdot \alpha 1 = \langle 1 - e^{-\{K(-Ln(1-\mu\alpha 1))^{\beta}\}^{\frac{1}{\beta}}}, e^{-\{K(-Ln(\eta\alpha 1))^{\beta}\}^{\frac{1}{\beta}}}, e^{-\{K(-Ln(\nu\alpha 1))^{\beta}\}^{\frac{1}{\beta}}} \rangle;$

4. $\alpha^K = \langle e^{-\{K(-Ln(\mu\alpha_1))^\beta\}^{\frac{1}{\beta}}}, 1 - e^{-\{K(-Ln(1-\eta\alpha_1))^\beta\}^{\frac{1}{\beta}}}, 1 - e^{-\{K(-Ln(1-\nu\alpha_1))^\beta\}^{\frac{1}{\beta}}} \rangle.$

Definition 8. *Let* $\alpha = \{\langle x, \mu\alpha(x), \nu\alpha(x), \eta\alpha(x)|x \in X\rangle\}$ $\alpha_1 = \{\langle x, \mu\alpha_1(x), \nu\alpha_1(x), \eta\alpha_1(x)|x \in X\rangle\}$ *and* $\alpha_2 = \{\langle x, \mu\alpha_2(x), \nu\alpha_2(x), \eta\alpha_2(x)|x \in X\rangle\}$ *be any three PFS, and their set operators are defined as*

1. $\alpha_1 \subseteq \alpha_2 \Leftrightarrow \mu\alpha_1(x) \le \mu\alpha_2(x), \nu\alpha_1(x) \le \nu\alpha_2(x)$ *and* $\eta\alpha_1(x) \ge \eta\alpha_2(x) \forall x \in X$;
2. $\alpha_1 \cup \alpha_2 = \{\langle x, \{OA\{\mu\alpha_1(x), \mu\alpha_2(x)\}\}, \{Eg_A\{\nu\alpha_1(x), \nu\alpha_2(x)\}\}, \{Eg_A\{\eta\alpha_1(x), \eta\alpha_2\}(x)\}|x \in X\rangle\}$;
3. $\alpha_1 \cap \alpha_2 = \{\langle x, \{Eg_A\{\mu\alpha_1(x), \mu\alpha_2(x)\}\}, \{OA\{\nu\alpha_1(x), \nu\alpha_2(x)\}\}, \{OA\{\eta\alpha_1(x), \eta\alpha_2(x)\}\}|x \in X\}\rangle\}$;
4. $\alpha^c = \{\langle x, \eta\alpha(x), \nu\alpha(x), \mu\alpha(x)|x \in X\rangle\}$.

3 Proposed PF Aczel-Alsina power aggregation operator

The current section defines a series of PF Aczel-Alsina power geometric operators that incorporate CLs with the evaluated PFNs.

3.1 PF Aczel-Alsina power geometric aggregation operator

In this part, we built the PF weighted and ordered weighted Aczel-Alsina power geometric AOs. Additionally, we investigate several fundamental aspects of these proposed operators.

3.1.1 PF Aczel-Alsina power weighted geometric aggregation operator

By employing the fundamental operations of AA aggregation tools, we derived appropriate methodologies, includ- ing PFAAPWG, with reliable properties while considering PFNs. Additionally, we applied a weighted suppport degree throughout our article, using the following equation: $\chi_j = \frac{w_j(1+\breve{U}(\alpha_j))}{\sum_{j=1}^{\rho} w_j(1+\breve{U}(\alpha_j))}$ where the suppport of α_j is denoted by $\breve{U}(\alpha_j) = \sum_{j=1, h\neq j}^{\rho} supp(\alpha_j, \alpha_h), j = 1, 2, \ldots, \rho, h = 1, 2, \ldots, r$ and the associated weight vector of α_j is $w = (w_1, w_2, \ldots, w_\rho,)^T, j = 1, 2, \ldots, \rho, w_j > 0,$ *and* $\sum_{j=1}^{\rho} w_j = 1.$

Definition 9. *Let* $\alpha_j = (\mu_j, \eta_j, \nu_j)(j = 1, 2, \ldots, \rho)$ *be a set of PFNs and* $\chi = (\chi_1, \chi_2, \ldots, \chi_\rho)^T$ *be the weight vectors for PFNs with the condition* $\sum_{j=1}^{\rho} \chi_j = 1$. *Then, the mapping PFAAPWG:* $b^\rho \to b$ *operator is given as follows: PFAAPWG* $\{\alpha_1, \alpha_2, \ldots, \alpha_\rho\}$

$$= \bigoplus_{j=1}^{\rho} (\alpha_j)^{\chi_j} = (\alpha_1)^{\chi_1} \oplus (\alpha_2)^{\chi_2} \oplus \cdots \oplus (\alpha_\rho)^{\chi_\rho}. \quad (3)$$

Theorem 1. *The aggregated value of the PFNs* α_j *for* $j = 1, 2, \ldots, \rho$ *with respect to the weight vector* $\chi = (\chi_1, \chi_2, \ldots, \chi_\rho)^T$ *obtained using the PFAAPWG Equation 5 is also a PFN and is given by PFAAPWG* $\{\alpha_1, \alpha_2, \ldots, \alpha_\rho\} =$

$-\{\sum^{\rho} (\chi (-Ln(\mu))^\alpha)\}^{\frac{1}{\alpha}}$ $\quad -\{\sum^{\rho} (\chi (-Ln(1-\eta$

$$= \left\langle e^{-\{\sum_{j=1}^{\rho}(\chi_j(-Ln(\mu_{\alpha_j}))^{\alpha})\}^{\frac{1}{\alpha}}},\ 1-e^{-\{\sum_{j=1}^{\rho}(\chi_j(-Ln(1-\eta_{\alpha_j}))^{\alpha})\}^{\frac{1}{\alpha}}},\ 1-e^{-\{\sum_{j=1}^{\rho}(\chi_j(-Ln(1-\nu_{\alpha_j}))^{\alpha})\}^{\frac{1}{\alpha}}}\right\rangle \quad (4)$$

Proof. By mathematical induction the proof as follows:

1. For $\rho = 2$, we have PFAAPWG $((\alpha_1, \alpha_2)) = \alpha_1^{\chi_1} \oplus \alpha_2^{\chi_2}$. By operational laws, we get $\alpha_1^{\chi_1} = \langle e^{-\{(\chi_1(-Ln(\mu_{\alpha_1}))^{\alpha}\}^{\frac{1}{\alpha}}}, 1-e^{-\{(\chi_1(-Ln(1-\eta_{\alpha_1}))^{\alpha}\}^{\frac{1}{\alpha}}}, 1-e^{-\{(\chi_1(-Ln(1-\nu_{\alpha_1}))^{\alpha}\}^{\frac{1}{\alpha}}}\rangle$. analogously, for $\alpha_2^{\chi_2} = \langle e^{-\{(\chi_2(-Ln(\mu_{\alpha_2}))^{\alpha}\}^{\frac{1}{\alpha}}}, 1-e^{-\{(\chi_2(-Ln(1-\eta_{\alpha_2}))^{\alpha}\}^{\frac{1}{\alpha}}}, 1-e^{-\{(\chi_2(-Ln(1-\nu_{\alpha_2}))^{\alpha}\}^{\frac{1}{\alpha}}}\rangle$. PFAAPWG $(\alpha_1, \alpha_2) = (\alpha_1^{\chi_1} \oplus \alpha_2^{\chi_2}) = \langle e^{-\{(\chi_1(-Ln(\mu_{\alpha_1}))^{\alpha}\}^{\frac{1}{\alpha}}}, 1-e^{-\{(\chi_1(-Ln(1-\eta_{\alpha_1}))^{\alpha}\}^{\frac{1}{\alpha}}}, 1-e^{-\{(\chi_1(-Ln(1-\nu_{\alpha_1}))^{\alpha}\}^{\frac{1}{\alpha}}}\rangle \oplus \langle e^{-\{(\chi_2(-Ln(\mu_{\alpha_2}))^{\alpha}\}^{\frac{1}{\alpha}}}, 1-e^{-\{(\chi_2(-Ln(1-\eta_{\alpha_2}))^{\alpha}\}^{\frac{1}{\alpha}}}, 1-e^{-\{(\chi_2(-Ln(1-\nu_{\alpha_2}))^{\alpha}\}^{\frac{1}{\alpha}}}\rangle = \langle e^{-\{(\chi_1(-Ln(\mu_{\alpha_1}))^{\alpha}+(\chi_2(-Ln(\mu_{\alpha_2}))^{\alpha}\}^{\frac{1}{\alpha}}}, 1-e^{-\{(\chi_1(-Ln(1-\eta_{\alpha_1}))^{\alpha}+(\chi_2(-Ln(1-\eta_{\alpha_2}))^{\alpha}\}^{\frac{1}{\alpha}}}, 1-e^{-\{(\chi_1(-Ln(1-\nu_{\alpha_1}))^{\alpha}+(\chi_2(-Ln(1-\nu_{\alpha_2}))^{\alpha}\}^{\frac{1}{\alpha}}}\rangle$
$$= \left\langle e^{-\{\sum_{j=1}^{2}(\chi_j(-Ln(\mu_{\alpha_j}))^{\alpha})\}^{\frac{1}{\alpha}}},\ 1-e^{-\{\sum_{j=1}^{2}(\chi_j(-Ln(1-\eta_{\alpha_j}))^{\alpha})\}^{\frac{1}{\alpha}}},\ 1-e^{-\{\sum_{j=1}^{2}(\chi_j(-Ln(1-\nu_{\alpha_j}))^{\alpha})\}^{\frac{1}{\alpha}}}\right\rangle.$$
Hence, this true for j=2.
2. Now, supppose that this will be true for j=k. Then we have the following equation: PFAAPWG $\{(\alpha_1\alpha_2, \ldots, \alpha_k)\}$ =
$$\left\langle e^{-\{\sum_{j=1}^{k}(\chi_k(-Ln(\mu_{\alpha_k}))^{\alpha}\}^{\frac{1}{\alpha}}},\ 1-e^{-\{\sum_{j=1}^{k}(\chi_k(-Ln(1-\eta_{\alpha_k}))^{\alpha}\}^{\frac{1}{\alpha}}},\ 1-e^{k} \ldots \right.$$

$(\chi_k (-Ln(1-\nu_{\alpha_k}))^{\alpha})\}^{\frac{1}{\alpha}}\rangle$. Now, we have to show that it also hods for j=k+1 as follows

$$PFAAPWG\{(\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1})\} = \langle e^{-\{\sum_{j=1}^{k}(\chi_{\alpha_k}(-Ln(\mu_{\alpha_k}))^{\alpha})\}^{\frac{1}{\alpha}}}, 1-e^{-\{\sum_{j=1}^{k}(\chi_{\alpha_k}(-Ln(1-\eta_{\alpha_k}))^{\alpha})\}^{\frac{1}{\alpha}}}, 1-e^{-\{\sum_{j=1}^{k}(\chi_{\alpha_k}(-Ln(1-\nu_{\alpha_k}))^{\alpha})\}^{\frac{1}{\alpha}}}\rangle \oplus$$

$$\langle e^{-\{\sum_{j=k+1}^{k+1}(\chi_{\alpha_{k+1}}(-Ln(\mu_{\alpha_{k+1}}))^{\alpha})\}^{\frac{1}{\alpha}}}, 1-e^{-\{\sum_{j=k+1}^{k+1}(\chi_{\alpha_{k+1}}(-Ln(1-\eta_{\alpha_{k+1}}))^{\alpha})\}^{\frac{1}{\alpha}}}, 1-e^{-\{\sum_{j=k+1}^{k+1}(\chi_{\alpha_{k+1}}(-Ln(1-\nu_{\alpha_{k+1}}))^{\alpha})\}^{\frac{1}{\alpha}}}\rangle = \langle e^{-\{\sum_{j=1}^{k+1}(\chi_{\alpha_j}(-Ln(\mu_{\alpha_j}))^{\alpha})\}^{\frac{1}{\alpha}}}, 1-e^{-\{\sum_{j=1}^{k+1}(\chi_{\alpha_j}(-Ln(1-\eta_{\alpha_j}))^{\alpha})\}^{\frac{1}{\alpha}}}, 1-e^{-\{\sum_{j=1}^{k+1}(\chi_{\alpha_j}(-Ln(1-\nu_{\alpha_j}))^{\alpha})\}^{\frac{1}{\alpha}}}\rangle$$

which is true for j=k+1. □

property 1. *The PFAAPWG is idempotent. i.e., If $\alpha_j = \alpha$ for all j, then PFAAPWG $(\alpha_1, \alpha_2, \ldots, \alpha_\rho) = \alpha$.*

property 2. *The PFAAPWG is boundedness. i.e., For a collection of PFNs α_j for all $j = 1, 2, \ldots, \rho$ and*

$\alpha^- = min(\alpha_1, \alpha_2, \ldots, \alpha_\rho)$ *and* $\alpha^+ = max(\alpha_1, \alpha_2, \ldots, \alpha_\rho)$. *Then* $\alpha^- \leq PFAAPWAA(\alpha_1, \alpha_2, \ldots, \alpha_\rho) \leq \alpha^+$.

property 3. *The PFAAPWG is monotonicity. i.e., for any two PFNs* $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \rangle$ *and* $\alpha'_j = \langle \mu'_{\alpha_j}, \eta'_{\alpha_j}, \nu'_{\alpha_j} \rangle$ *such that* $\alpha_j \leq \alpha'_j$ *for all* $j = 1, 2, \ldots, \rho$. *Then* $PFAAPWG(\alpha_1, \alpha_2, \ldots, \alpha_\rho) \leq PFAAPWG(\alpha'_1, \alpha'_2, \ldots, \alpha'_\rho)$.

3.1.2 PF Aczel-Alsina power ordered weighted geometric aggregation operator

In this part, a novel PFAAPOWG. This operator considers the ordered weights associated with the aggregated PFNs.

Definition 10. *Let $\alpha_j = (\mu_j, \eta_j, \nu_j)(j = 1, 2, \ldots, \rho)$ be a set of PFNs and $\chi = (\chi_1, \chi_2, \ldots, \chi_\rho)^T$ be the weight vectors for PFNs with the condition $\sum_{j=1}^{\rho} \chi_j = 1$. Then, the mapping PFAAPOWG: $b^\rho \rightarrow b$ operator is given as follows:* PFAAPOWG $\{\alpha_1, \alpha_2, \ldots, \alpha_\rho\} = \bigoplus_{j=1}^{\rho} (\alpha_j)^{\chi_j}$

$$= (\alpha_1)^{\chi_1} \oplus (\alpha_2)^{\chi_2} \oplus \cdots \oplus (\alpha_\rho)^{\chi_\rho}. \quad (5)$$

Theorem 2. *The aggregated value of the PFNs α_j for $j = 1, 2, \ldots, \rho$ with respect to the weight vector $\chi = (\chi_1, \chi_2, \ldots, \chi_\rho)^T$ obtained using the PFAAPOWG Equation 5 is also a PFN and is given by PFAAPOWG $\{\alpha_1, \alpha_2, \ldots, \alpha_\rho\}$ =*

$$= \left\langle e^{-\left\{\sum_{j=1}^{\rho}\left(\chi_j\left(-Ln\left(\mu_{\alpha_j}\right)\right)^{\alpha}\right)\right\}^{\frac{1}{\alpha}}}, 1 - e^{-\left\{\sum_{j=1}^{\rho}\left(\chi_j\left(-Ln\left(1-\eta_{\alpha_j}\right)\right)^{\alpha}\right)\right\}^{\frac{1}{\alpha}}}, 1 - e^{-\left\{\sum_{j=1}^{\rho}\left(\chi_j\left(-Ln\left(1-\nu_{\alpha_j}\right)\right)^{\alpha}\right)\right\}^{\frac{1}{\alpha}}} \right\rangle \quad (6)$$

property 4. *The PFAAPOWG is idempotent. i.e., If $\alpha_j = \alpha$ for all j, then PFAAPOWG $(\alpha_1, \alpha_2, \ldots, \alpha_\rho) = \alpha$.*

property 5. *The PFAAPOWG is boundedness. i.e., For a collection of PFNs α_j for all $j = 1, 2, \ldots, \rho$ and $\alpha^- = min(\alpha_1, \alpha_2, \ldots, \alpha_\rho)$ and $\alpha^+ = max(\alpha_1, \alpha_2, \ldots, \alpha_\rho)$. Then $\alpha^- \leq PFAAPWAA(\alpha_1, \alpha_2, \ldots, \alpha_\rho) \leq \alpha^+$.*

property 6. *The PFAAPOWG is monotonicity. i.e., for any two PFNs $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \rangle$ and $\alpha'_j = \langle \mu'_{\alpha_j}, \eta'_{\alpha_j}, \nu'_{\alpha_j} \rangle$ such that $\alpha_j \leq \alpha'_j$ for all $j = 1, 2, \ldots, \rho$. Then $PFAAPOWG(\alpha_1, \alpha_2, \ldots, \alpha_\rho) \leq PFAAPOWG(\alpha'_1, \alpha'_2, \ldots, \alpha'_\rho)$.*

4 Evaluation of PFMCDM using proposed operators

This part illustrates how the proposed operators are applied by solving an PFMCDM model.

4.1 Procedure for PFMCDM problems

This section, presents a procedure for solving PFMCDM problems using the proposed operators.

Step 1 Let $A = (A_1, A_2, \ldots, A_\kappa)$ be a finite number of alternatives, and $\alpha = (B_1, B_2, \ldots, B_\rho)$ be the set of criteria.

Let $w = (w_1, w_2, \ldots, w_\rho)^T$ be the weight vector of criteria, where $w_j \geq 0$, $j = 1,2,\ldots,\rho$ such that $\sum_{j=1}^{\rho} w_j = 1$.

The PF decision matrix $D = [A_{ij}]_{\kappa\times\rho}$ evaluates the alternatives under each criteria, where $\mu_{ij}, \eta_{ij}, \nu_{ij}$ indicates the truth, falsity and indeterminacy membership function respectively.

Step 2 To standardize an PF decision matrix that includes a cost type criteria, apply the following Equation 7

$$D = [A_{ij}]_{\kappa\times\rho} = \langle \mu_{ij}, \eta_{ij}, \nu_{ij} \rangle \text{if benefit type}$$

$\langle \nu_{ij}, \eta_{ij}, \mu_{ij} \rangle$ if cost type (7)

This section demonstrates the application of the proposed operators through the resolution of an PFMCDM model.

Step 3 Apply the proposed operators to aggregate the evaluations of each criteria across all alternatives.

Step 4 Determine the optimal alternative by ranking the options based on their aggregated score values.

5 Numerical illustration

Let's consider a practical example of a MCDM problem, which addresses the selection of food suppliers within food quality management. In this example, five suppliers, denoted as alternatives A_i ($i = 1, 2, 3, 4, 5$), are eval- uated based on four distinct criteria B_j ($j = 1, 2, 3, 4$): product quality, delivery reliability, cost efficiency, and sustainability practices. These evaluations are weighted according to the vector: $\tau = (0.2, 0.1, 0.3, 0.4)^T$. This structure allows decision-makers to systematically evaluate each supplier's performance against multiple relevant factors, ensuring a balanced and well-informed choice. **Step 1**: The decision matrix for this MCDM problem, featuring -induced PF preference values as assessed by a decision expert, is provided in Table 1. **Step 2** Since all the criteria are beneficial, normalization of the PF decision matrix is not required.

Step 3 Combine all the criteria, each associated with its own unique PF preference value for each alternative, us- ing the PFAAPWG in Equation 6 to obtain the overall PF. α_i of the corresponding A_i as $\alpha_1 = \langle 0.57, 0.234, 0.097\rangle$, $\alpha_2 = \langle 0.701, 0.121, 0.18\rangle$, $\alpha_3 = \langle 0.649, 0.194, 0.114\rangle$, $\alpha_4 = \langle 0.621, 0.183, 0.127\rangle$, $\alpha_5 = \langle 0.726, 0.308, 0.0969\rangle$. Com-
bine all the criteria, each with its own distinct PF preference value for each alternative using PFAAPOWG Equa- tion **??** to get the overall PF α_i of the corresponding A_i as $\alpha_1 = \langle 0.561, 0.261, 0.084\rangle$, $\alpha_2 = \langle 0.687, 0.116, 0.202\rangle$, $\alpha_3 = \langle 0.661, 0.219, 0.109\rangle$, $\alpha_4 = \langle 0.628, 0.203, 0.130\rangle$, $\alpha_5 = \langle 0.73, 0.339, 0.085\rangle$.

Step 4 Calculate the score values corresponding to the PF α_i obtained in Step 3 based on the PFAAPWG respec- tively are $S(\alpha_1) = 0.242$, $S(\alpha_2) = 0.4001$, $S(\alpha_3) = 0.341$, $S(\alpha_4) = 0.311$, $S(\alpha_5) = 0.321$. Based on PFAAPWG,

Table 1: PF decision matrix

	C_1	C_2	C_3	C_4
A_1	⟨0.5, 0.04, 0.1⟩	⟨0.7, 0.12, 0.02⟩	⟨0.3, 0.5, 0.1⟩	⟨0.4, 0.2, 0.2⟩
A_2	⟨0.6, 0.2, 0.1⟩	⟨0.5, 0.2, 0.09⟩	⟨0.5, 0.02, 0.4⟩	⟨0.7, 0.2, 0.01⟩
A_3	⟨0.5, 0.2, 0.2⟩	⟨0.2, 0.4, 0.06⟩	⟨0.5, 0.3, 0.1⟩	⟨0.55, 0.23, 0.2⟩
A_4	⟨0.6, 0.01, 0.2⟩	⟨0.52, 0.1, 0.23⟩	⟨0.3, 0.4, 0.2⟩	⟨0.5, 0.2, 0.15⟩
A_5	⟨0.5, 0.3, 0.11⟩	⟨0.2, 0.3, 0.1⟩	⟨0.05, 0.6, 0.1⟩	⟨0.3, 0.24, 0.2⟩

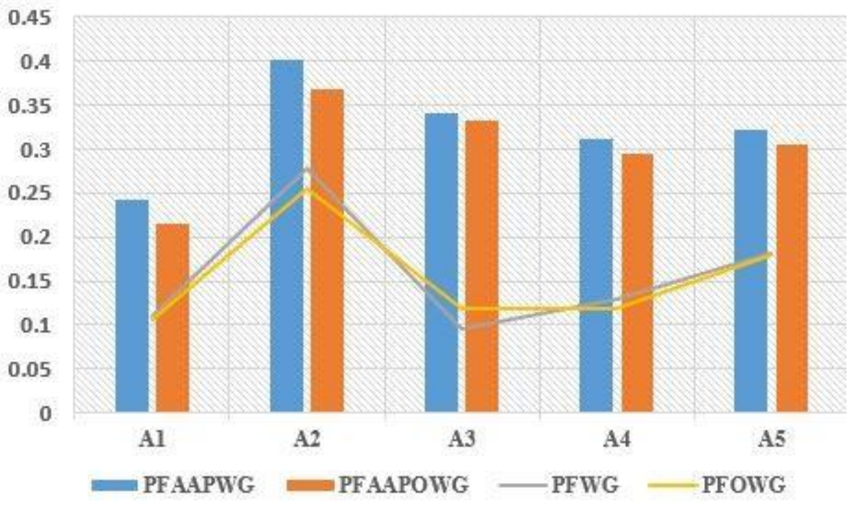

Figure 2: Graphical comparison of proposed and existing operators

the score value .Thus, we have $A2 > A3 > A5 > A4 > A1$. Hence the best alternative is $A2$.
Calculate the score values corresponding to the PF αi obtained in Step 3 based on the PFAAPOWG respectively are $S(\alpha 1) = 0.216$, $S(\alpha 2) = 0.369$, $S(\alpha 3) = 0.333$, $S(\alpha 4) = 0.294$, $S(\alpha 5) = 0.306$. Based on PFAAPOWG, the score value .Thus, we have $A2 > A3 > A5 > A4 > A1$. Hence the best alternative is $A2$.

6 Comparative analysis

In this discussion, we compare the overall ranking results obtained using the proposed PFAAPWG and PFAAPOWG operators for the example in Section 5, with the existing results based on the PF weighted geometric aggregation operator (PFWG) and PF orered weighted geometric aggregation operator (PFOWG). From Table 2,we observe that the best alternatives of the proposed operators are same as that of the existing operators. This comparison is visually represented in Figure 2.

Table 2: Comparison of the existing operators with the proposed operators

Method	Operator	Ranking	Best
Wei, G. (2017)	PFWG	$A_2 > A_5 > A_4 > A_1 > A_3$	A_2
Proposed operator	PFOWG	$A_2 > A_5 > A_3 > A_4 > A_1$	A_2
	PFAAPWG	$A_2 > A_3 > A_5 > A_4 > A_1$	A_2
	PFAAPOWG	$A_2 > A_3 > A_5 > A_4 > A_1$	A_2

7 Conclusion

This paper introduces the PFAAPG aggregation framework, specifically focusing on the development of the PFAAPWG and PFAAPOWG operators. The fundamental mathematical properties of these operators are sys- tematically analyzed to establish their theoretical foundation. A key distinguishing feature of these operators is their ability to not only incorporate the evaluated arguments provided by decision experts but also to account for the significance levels associated with their assessments, thereby enhancing the robustness and adaptability of the decision-making process. To demonstrate the practical applicability of the proposed operators, a PFMCDM approach is developed and applied to a real-world food industry selection problem, where four evaluation criteria are considered. The effectiveness and reliability of this approach are validated through a comparative analysis with existing aggregation operators, including the PFOWG, and PFAAPWG operators. Furthermore, graphical representations are provided to enhance interpretability, offering a more intuitive visualization of the decision- making outcomes. Future research directions include extending this approach by developing advanced picture fuzzy geometric aggregation operators and investigating the integration of probabilistic information, uncertainty modeling, and additional influencing factors to further refine decision-making frameworks. These enhancements aim to improve the precision and applicability of picture fuzzy aggregation methods in complex multi-criteria decision-making environments.

References

Atanassov, K. T. (1986). Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, *20*(1), 87-96.

Ates, F., & Akay, D. (2020). Some picture fuzzy Bonferroni mean operators with their application to multicriteria decision making, *International Journal of Intelligent Systems*, *35*(4), 625-649.

Biswas, A., & Deb, N. (2021). Pythagorean fuzzy Dombi power aggregation operators for solving multi-attribute decision making problems, *Granular Computing*, *6*, 991–1007.

Cuong, B. C., & Kreinovich, V. (2013). Picture fuzzy sets-a new concept for computational intelligence problems.
In 2013 third world congress on information and communication tech- nologies (WICT 2013), 1-6. IEEE.

Cuong, B. C., Kreinovitch, V., & Ngan, R. T. (2016). A classification of representable t-norm operators for picture fuzzy sets. *In 2016 eighth international conference on knowledge and systems engineering (KSE)*, 19-24. IEEE.

Garg, H. (2017). Some picture fuzzy aggregation operators and their applications to multicriteria decision-making.
, *Arabian Journal for Science and Engineering*, *42*(12), 5275-5290.

Garg, H., Tiwari, P., & Prasad, V. (2023). Dombi prioritized aggregation operators for intuitionistic fuzzy infor- mation and their application in multi-attribute decision making, *Alexandria Engineering Journal*, *67*, 229–240.

Hussain, A., Latif, S., & Ullah, K. (2022). A novel approach of picture fuzzy sets with unknown degree of weights based on Dombi aggregation operators, *Journal of Innovative Research in Mathematical and Computational Sciences*, *1*(2), 18–39.

Jana, C., & Pal, M. (2019). Assessment of enterprise performance based on picture fuzzy Hamacher aggregation operators, *Symmetry*, *11*(1), 75.

Jana chiranjibe, Tapan Senapati, Madhumangal Pal, and Ronald R. Yager. Picture fuzzy Dombi aggregation operators: application to MADM process, *Applied Soft Computing*, *74*, 99-109.

Khan, S., Abdullah, S., & Ashraf, S. (2019). Picture fuzzy aggregation information based on Einstein operations and their application in decision making,

Mathematical Sciences, *13*(3), 213-229.
Khan, Q., Riaz, M., & Latif, S. (2022). Multiple attribute group decision making based on intuitionistic fuzzy Dombi generalized power aggregation operators, *Mathematical Problems in Engineering*, *2022*(1), 4634411.
Lin, M., Li, X., Chen, R., Fujita, H., & Lin, J. (2022). Picture fuzzy interactional partitioned Heronian mean aggregation operators: an application to MCDM process. *Artificial Intelligence Review*, 1-38.
Liu, P. D., & Liu, X. (2017). Multiattribute group decision-making methods based on linguistic intuitionistic fuzzy power Bonferroni mean operators. *Complexity, 3571459* (1), 1-15.
Ma, L., Zhang, T., & Wang, Y. (2024). Decision algorithm for q-rung orthopair fuzzy information based on Dombi aggregation operators with applications in agricultural systems, *IEEE Access*, *12*, 25762–25778.
Qin, Y., Cui, X., Huang, M., Zhong, Y., Tang, Z., & Shi, P. (2021). Multiple-attribute decision-making based on picture fuzzy Archimedean power Maclaurin symmetric mean operators. *Granular Computing*, *6*(3), 737-761.
Qiyas, M., Khan, M. A., Khan, S., & Abdullah, S. (2020). Concept of Yager operators with the picture fuzzy set environment and its application to emergency program selection. *International Journal of Intelligent Computing and Cybernetics*, *13*(4), 455-483.
Rahman, K., Ayub, S., & Abdullah, S. (2021). Generalized intuitionistic fuzzy aggregation operators based on levels for group decision making. *Granular Computing*, *6*, 867-886.
Seikh, M. R., & Mandal, U. (2021). Some Picture Fuzzy Aggregation Operators based on Frank t-norm and t-conorm: Application to MCDM Process. *Informatica*, *45*(3).
Senapati, T. (2022). Approaches to multi-attribute decision-making based on picture fuzzy Aczel–Alsina geometric aggregation operators. *Computational and Applied Mathematics*, *41*(1), 40.
Ullah, K. (2021). Picture fuzzy maclaurin symmetric mean operators and their applications in solving multiat- tribute decision-making problems. *Mathematical Problems in Engineering*, 1-13.
Ullah, K., Naeem, M., Hussain, A., Waqas, M., & Haleemzai, I. (2023). Evaluation of electric motor cars based frank power aggregation operators under picture fuzzy information and a multi-attribute group decision-making process. *IEEE Access*, *11*, 67201-67219.
Wang, R., Wang, J., Gao, H., & Wei, G. (2018). Methods for MCDM with picture fuzzy muirhead mean operators and their application for evaluating the financial investment risk, *Symmetry*, *11*(1), 6.
Wei, G. (2017). Picture fuzzy aggregation operators and their application to multiple attribute decision making,
Journal of Intelligent & Fuzzy Systems, *33*(2), 713-724.
Wei, G. (2018). Picture fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. , *Fundamenta Informaticae*, *157*(3), 271-320.
Wei, D., Liu, X., & Ma, Z. (2022). Fermatean fuzzy Dombi operators and BWM-entropy-based combined com- promise solution approach: an application to green supplier selection, *Entropy*, *24*(6), 776.
Yager, R. R. (1988). On ordered weighted geometric aggregation operators in multi criteria decision making. *IEEE Transactions on systems Man and Cybernetics*, *18(1)*(1), 183-190.
Yager, R. R (2001). The power geometric operator. *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, *31* (6), 724-731.
Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.

Zhang, H., Zhang, R., Huang, H., & Wang, J. (2018). Some picture fuzzy Dombi Heronian mean operators with their application to multi-attribute decision-making, *Symmetry*, *10*(11), 593.

Naeem, M. Khan, Y. Ashraf, S. Weera, W. and Batool, B. (2022). A novel picture fuzzy Aczel-Alsina geometric aggregation information: Application to determining the factors affecting mango crops, *AIMS Math*, *7* (7), 12264-12288.