The Chromatic Number of Certain Families of Fuzzy Graphs

M.G. Karunambigai[1], Jessalet Ann Mathew[2]*

[1,2*]Department of Mathematics, Sri Vasavi College, Erode-638 316, Tamilnadu, India.
Email: [1]karunsvc@yahoo.in

***Corresponding author:** Jessalet Ann Mathew
*Email: jessaletannmathew@gmail.com.

Abstract
Fuzzy coloring of a fuzzy graph is one of the important areas of graph theory, and it plays an important role in real-life problems. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Fuzzy coloring is an assignment of basic or fuzzy colors to the vertices of G, and it is proper, (i) If two vertices are connected by a strong edge, then they either have different basic or fuzzy colors (if necessary), or one vertex can have a basic color and the other can have a fuzzy color corresponding to different basic color. (ii) If two vertices are connected by a weak edge, then they either have the same or different fuzzy colors, or one vertex can have a basic color and the other can have a fuzzy color corresponding to the same basic color. The minimum number of colors (basic or fuzzy) needed for a proper fuzzy coloring of G is called the chromatic number of G and is denoted by $\chi_f(G)$. In this paper, the chromatic number of certain families of fuzzy graphs, such as path, cycle, star, wheel, and complete graphs are derived. Some relevant properties on fuzzy coloring of path, cycle, star, wheel, and complete graphs are proved. Furthermore, an application on fuzzy coloring is formulated using the chromatic number of G.

Keywords: Fuzzy graph, Strong edge, Weak edge, Fuzzy coloring, Chromatic number.

1. Introduction

Fuzzy graph coloring is one of the most extensively researched topics in combinatorial optimization[1] and it plays a vital role in addressing uncertainty, imprecision, and ambiguity across diverse fields. One of the most practical problems in the literature was the traffic light problem[2], which was resolved by applying the crisp graph coloring method. However, in the traffic light problem, certain roads are busier than others. Additionally, sometimes two roads may occasionally be opened concurrently with caution. Both "busy" and "cautious" are fuzzy terms here. In 2005, Susana Munoz et al.[3] introduced the coloring of fuzzy graphs, and they designed the traffic light problem using fuzzy graphs. In that paper, Susana Munoz et al. proposed a method for coloring the vertices of fuzzy graphs with a crisp vertex set and a fuzzy edge set (the type 1 fuzzy graphs).

Furthermore, in 2006, Eslahchi and Onagh[4] developed a similar method of coloring for fuzzy graphs with fuzzy vertex sets and fuzzy edge sets (the type 2 fuzzy graphs) based on strong adjacencies between vertices. In 2015, Sovan Samanta et al.[2] introduced a new concept to color a fuzzy graph by using fuzzy colors based on the strength of an edge incident to a vertex, and they also introduced the fuzzy chromatic number, which motivated us to develop an extension for the procedure of coloring a fuzzy graph using fuzzy colors based on the strength of an edge incident to a vertex. In this paper, the chromatic number of certain families of fuzzy graphs, such as path, cycle, star, wheel, and complete graphs, has been found, and some properties on fuzzy coloring are given.

The structure of this article is as follows : In Section 1, the introduction to the fuzzy coloring of a fuzzy graph is given. In Section 2, we review the fundamental concepts of fuzzy graph theory and fuzzy coloring that are essential for our research. In Section 3, we discuss the key concepts essential for fuzzy coloring and introduce an improved procedure for the fuzzy coloring of fuzzy graphs using fuzzy colors. We also illustrate the fuzzy coloring of a fuzzy graph in three different cases : when all edges are strong, when all edges are weak, and when the graph contains both weak and strong edges. Furthermore, we provide the chromatic number for each such fuzzy graph. In Section 4, the chromatic numbers of certain families of fuzzy graphs are found by using fuzzy colors based on the strength of an edge incident to a vertex. In Section 5, an application of fuzzy coloring is illustrated. We examine the literacy rates of various states in India and explore the relationships between these states with the goal of improving literacy. Finally, conclusions are given in Section 6.

2. Preliminaries

This section begins with a review of some definitions from fuzzy graph theory and fuzzy coloring, which helps to find the chromatic number of fuzzy graphs.

Let $G = (V, E)$ be a graph consisting of a non-empty finite set V of elements called *vertices* and a finite set E of ordered pairs of distinct vertices called *edges*. An edge (v_i, v_j) is said to be incident to the vertices v_i and v_j.

Definition 2.1. [5] Let $G = (V, E)$ be a graph. Chromatic number of G is the minimum number of colors needed to color the vertices of G, such that no two adjacent vertices can have the same color (proper coloring) and is denoted by $\chi(G)$.

Theorem 2.1. [6] *Let G be a trivial graph, then* $\chi(G) = 1$.

Theorem 2.2. [2] *Let P_n be a path, then* $\chi(P_n) = 2$.

Theorem 2.3. [5] *Let C_n be a cycle, then*

$$\chi(C_n) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Theorem 2.4. [6] *Let S_n be a star, then* $\chi(S_n) = 2$.

Theorem 2.5. [6] *Let $W_n, n \geq 3$ be a wheel graph, then*

$$\chi(W_n) = \begin{cases} 4 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Theorem 2.6. [6] *Let K_n be a complete graph, then* $\chi(K_n) = n$.

Theorem 2.7. [7] *The complete graph K_n has a Hamiltonian decomposition for all n.*

Definition 2.2. (*fuzzy set, membership function* [8]) Let X be universe of discourse, then a fuzzy set A in X is a set of ordered pairs : $A = \{(x, \mu_A(x)) | x \in X\}$, where $\mu_A(x) : X \to [0, 1]$ is called the membership function (generalized characteristic function). i.e., membership function assigns a fuzzy index $\mu_A(x)$ to every member x of a fuzzy set A in the interval of $[0,1]$. Which is often called membership value of x in A.

Definition 2.3. (*fuzzy graph* [9]) A fuzzy graph $G = (V, \sigma, \mu)$ is a pair of functions (σ, μ), where $\sigma : V \to [0, 1]$ is a fuzzy subset of a non empty set V, and $\mu : V \to [0, 1]$ is a symmetric fuzzy relation on σ, such that the relation $\mu(v_i, v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$ is satisfied for all $v_i, v_j \in V$ and $(v_i, v_j) \in E \subset V \times V$.
Here, $\sigma(v_i)$ denote the degree of membership of the vertex v_i, and $\mu(v_i, v_j)$ denotes the degree of membership of the edge relation $e_{ij} = (v_i, v_j)$ on $V \times V$.

Note : In this paper, we denote $\sigma(v_i) \wedge \sigma(v_j) = min\{\sigma(v_i), \sigma(v_j)\}$, and $\sigma(v_i) \vee \sigma(v_j) = max\{\sigma(v_i), \sigma(v_j)\}$.

Definition 2.4. (*fuzzy subgraph* [9]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The fuzzy graph $G_1 = (V, \sigma_1, \mu_1)$ is called a fuzzy subgraph of G if $\sigma_1(x) \le \sigma(x)$ for all x and $\mu_1(x, y) \le \mu(x, y)$ for all edges (x, y), $x, y \in V$.

Definition 2.5. (*trivial graph* [11]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph with underlying crisp graph $G^* = (V, \sigma^*, \mu^*)$, where $\sigma^* = \{x \in V \mid \sigma(x) > 0\}$ and $\mu^* = \{(x, y) \in V \times V \mid \mu(x, y) > 0\}$. Then G is called trivial if $|\sigma^*| = 1$.

Definition 2.6. (*fuzzy path* [12]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph with underlying crisp graph G^*. A fuzzy path P_n in G is a sequence of distinct vertices $v_0, v_1, \ldots, v_n$ such that $\mu(v_{i-1}, v_i) > 0, 1 \le i \le n$. Here $n \ge 1$ is called the length of the path P_n.

Definition 2.7. (*strength of a fuzzy path* [12]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph and P_n be a fuzzy path in G. The strength of a fuzzy path P_n, $s(P_n)$ is given by $\wedge_{i=1}^{n} \mu(v_{i-1}, v_i)$.

Definition 2.8. (*components* [13],[9]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Two vertices u and v of G are said to be connected if there is a $u - v$ path in G. The relation "connected" is an equivalence relation on $V(G)$. Let $V_1, V_2, \ldots, V_\omega$ be the equivalence classes. The fuzzy subgraphs $G[V_1], G[V_2], \ldots, G[V_\omega]$ are called the components of G. If $\omega = 1$, the fuzzy graph G is connected; otherwise, the fuzzy graph G is disconnected with $\omega \ge 2$ components.

Definition 2.9. (*strong edge, weak edge* [2]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph and an edge $e = (v_i, v_j) \in G$ is called **strong** if $\frac{1}{2}\{\sigma(v_i) \wedge \sigma(v_j)\} \le \mu(v_i, v_j)$ and it is called **weak** otherwise.

Definition 2.10. (*strength of an edge* [2]) Let $G = (V, \sigma, \mu)$ be a fuzzy graph and the **strength of an edge** $(v_i, v_j) \in G$ is denoted by,

$$I(v_i, v_j) = \frac{\mu(v_i, v_j)}{\sigma(v_i) \wedge \sigma(v_j)}.$$

Definition 2.11. (*fuzzy cycle* [12]) A fuzzy path P_n in which $v_0 = v_n$ and $n \ge 3$, then P_n is called a fuzzy cycle C_n of length n.

Definition 2.12. (*fuzzy star* [8]) A fuzzy graph $G = (V, \sigma, \mu)$ is a fuzzy star S_n, if there exists only one vertex $v_0 \in V$ such that $\mu(v_0, v_i) > 0$ and $\mu(v_i, v_j) = 0 \;\forall\; v_i, v_j \in V$.

Definition 2.13. *(fuzzy wheel* [8]) A fuzzy graph $G = (V, \sigma, \mu)$ is a fuzzy wheel W_n if $E(G) = \{(v_0, v_i) \mid \mu(v_0, v_i) > 0,\ i = 1, 2, \ldots, n-1\} \cup \{(v_1, v_{n-1}) \mid \mu(v_1, v_{n-1}) > 0\} \cup \{(v_{j,} v_{j+1} \mid \mu(v_j, v_{j+1}) > 0, j = 1, 2, \ldots, n-2\}$.

Definition 2.14. *(complete fuzzy graph* [9]) A fuzzy graph $G = (V, \sigma, \mu)$ is a complete fuzzy graph K_n, if $\mu(v_i, v_j) = \sigma(v_i) \wedge \sigma(v_j)$ for every $v_i, v_j \in V$.

Definition 2.15. *(fuzzy cut vertex* [9]) A vertex x in $G = (V, \sigma, \mu)$ is a fuzzy cut vertex, if removal of x reduces the strength of connectedness between some pair of vertices in $G - x$.

Definition 2.16. *(block* [9]) A maximal connected fuzzy subgraph of $G = (V, \sigma, \mu)$, which has no fuzzy cut vertices called a block of a fuzzy graph G. If G has no fuzzy cut vertex, then G itself is a block.

Definition 2.17. *(basic color, fuzzy color* [2]) Mixing of a color with white color dilutes the density of the color. Suppose a quantity of q (≤ 1) units of a color c_k is mixed with $1 - q$ units of white color, then the mixture is called a standard mixture of the color c_k. The resultant color is called a fuzzy color of the color c_k with membership value q whereas c_k is called basic color.

Definition 2.18. [2] Let $C = \{c_1, c_2, \ldots, c_n\}, n \geq 1$ be a set of basic colors. The fuzzy set (C, f) is called the set of fuzzy colors, where $f : C \to [0, 1]$ with $f(c_i)$, be the amount of the basic color c_i per unit of standard mixture (the membership value of the fuzzy color corresponding to the basic color c_i). That is, the color $c' = (c_i, f(c_i))$ is called the fuzzy color corresponding to the basic color c_i with membership value $f(c_i)$. Thus, a basic color is also a fuzzy color whose membership value is taken as $1. i.e., (C, 1)$.

Definition 2.19. *(union* [9]) Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Let $V = V_1 \cup V_2$ and let $E = \{uv \mid u, v \in V, uv \in E_1$ or $uv \in E_2$ or $uv \in E_1 \cap E_2\}$, then the union of G_1 and G_2 denoted by $G_1 \cup G_2 : (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is defined by

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) & if\ u \in V_1 - V_2, \\ \sigma_2(u) & if\ u \in V_2 - V_1, \\ \sigma_1(u) \vee \sigma_2(u) & if\ u \in V_1 - V_2. \end{cases}$$

and

$$(\mu_1 \cup \mu_2)(u) = \begin{cases} \mu_1(u, v) & if\ uv \in E_1 - E_2, \\ \mu_2(u, v) & if\ uv \in E_2 - E_1, \\ \mu_1(u, v) \vee \mu_2(u, v) & if\ u \in V_1 - V_2. \end{cases}$$

Theorem 2.8. [10] *Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two fuzzy graphs, the chromatic numbers of G_1 and G_2 be $\chi_f(G_1)$ and $\chi_f(G_2)$, respectively. If fuzzy graph $G(V, E)$ is the union of two fuzzy graphs G_1 and G_2, then the chromatic number of G satisfies $max\{\chi_f(G_1), \chi_f(G_2)\} \leq \chi_f(G) \leq \chi_f(G_1) + \chi_f(G_2)$.*

3. Fuzzy Coloring of a Fuzzy Graph

Definition 3.1. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Fuzzy coloring is an assignment of basic or fuzzy colors to the vertices of a fuzzy graph G and it is *proper*,

(i) if two vertices are connected by a strong edge, then they either have different basic or fuzzy colors(if necessary), or one vertex can have a basic color and the other can have a fuzzy color corresponding to different basic color.

(ii) if two vertices are connected by a weak edge, then they either have same or different fuzzy colors, or one vertex can have a basic color and other can have a fuzzy color corresponding to the same basic color.

Example 1.

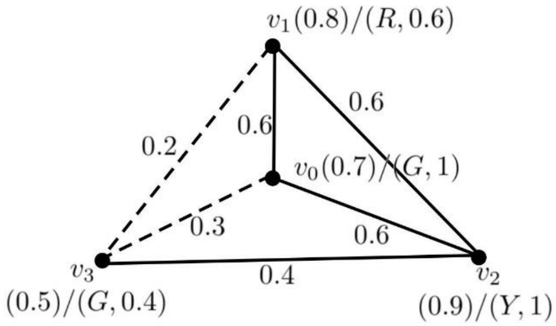

Figure 1. $\chi_f(G) = 3$.

Definition 3.2. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. *Perfect fuzzy coloring* (optimal fuzzy coloring) is an assignment of minimum number of colors (basic or fuzzy) for a proper fuzzy coloring of G. (refer Example 1).

Definition 3.3. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The minimum number of colors (basic or fuzzy) needed for a proper fuzzy coloring of G is called the *chromatic number* of G and is denoted by $\chi_f(G)$. (refer Example 1).

In a proper coloring of a crisp graph, two vertices have different colors if they are adjacent, and they may have the same color if they are not adjacent. But a proper fuzzy coloring of a fuzzy graph is based on the weak and strong edges of it. The procedure of proper fuzzy coloring of a fuzzy graph is based on the following concepts :

Case 3.1 : If an edge $e = (u, v) \in G$ is *strong,* then the vertices of e can be colored in any of the following ways.

(i) both the vertices of e can have different basic colors. $i.e.$, u and v can have colors $(R, 1)$ and $(Y, 1)$, respectively.

(ii) one vertex of e can have a basic color and other can have a fuzzy color corresponding to different basic color. $i.e.$, u and v can have colors $(R, 1)$ and $(Y, 0.a)$, respectively.

(iii) both the vertices of e can have different fuzzy colors (if necessary only). $i.e.$, u and v can have colors $(R, 0.a)$ and $(Y, 0.b)$, respectively.

Case 3.2 : If an edge $e = (u, v) \in G$ is *weak* then the vertices of e can be colored in any of the following ways.

(i) one vertex of e can have a basic color and other can have a fuzzy color corresponding to the same basic color.$i.e.$, u and v can have colors $(R, 1)$ and $(R, 0.a)$, respectively.

(ii) both the vertices of e can have the same fuzzy colors. $i.e.$, u and v can have colors $(R, 0.a)$ and $(R, 0.b)$, respectively.

(iii) both the vertices of e can have different fuzzy colors. i.e., u and v can have colors $(R, 0.a)$ and $(Y, 0.b)$, respectively.

Note : where a, b are positive integers.

3.1. Procedure of Proper Fuzzy Coloring of a Fuzzy Graph

Let $G = (V, \sigma, \mu)$ be a connected fuzzy graph and $C = \{c_1, c_2, \ldots, c_n\}, n \geq 1$ be a set of colors. In fuzzy graphs, there are two kinds of edges. i.e., *strong* and *weak* edges. It is important to note that a strong edge holds greater significance than a weak edge; in other words, there is a high correlation between the associated vertices. Based on these edges, the fuzzy coloring of a fuzzy graph is determined through three cases.

Case 3.1.1. *If all the edges are strong in a fuzzy graph.*
If all the edges in a fuzzy graph are strong, then the coloring of the fuzzy graph resembles the coloring of the crisp graph. i.e., if two vertices are connected by a strong edge, then they should be colored with two different basic colors.

Case 3.1.2. *If some edges are strong in a fuzzy graph.*
Let u_1 be a vertex and $N(u_1) = \{v_i, i = 1, 2, \ldots, n\}$ be the set of all neighborhoods of the vertex u_1. For simplicity, assume that v_1 is a vertex such that (v_1, u_1) is the sole strong edge incident to u_1, while all other edges $(u_1, v_i), i = 2, 3, \ldots, n$ incident to u_1 are weak. Now consider the vertex v_2 for coloring. There are three possibilities for coloring the vertex v_2.

Subcase 3.1.2.1. *Suppose all the neighboring vertices of v_2 are not colored.*
As the edge (v_1, u_1) is strong, color v_1 with $(c_1, 1)$ and color u_1 with $(c, 1)$. Since (u_1, v_2) is weak, v_2 can be colored with a fuzzy color corresponding to the color of u_1. Then the fuzzy color of v_2 be $(c, f(c))$, where $f(c)$ can be calculated as,

$$f(c) = 1 - I(u_1, v_2),$$

where,

$$I(u_1, v_2) = \frac{\mu(u_1, v_2)}{\sigma(u_1) \wedge \sigma(v_2)}.$$

Suppose, v_2 has some strong incident edges, say $(v_2, u_i), i = 2, \ldots, q$. Then color each vertex u_i with different basic colors.
Suppose an edge $(u_i, u_{i+1}) \in G$, where $i > 1$, is weak. Since (v_2, u_i) and (v_2, u_{i+1}) are strong, u_i and u_{i+1} must receive different basic colors. However, the edge (u_i, u_{i+1}) is weak. Therefore, vertex u_i will receive a basic color, and u_{i+1} will receive a fuzzy color corresponding to the color of vertex u_i with a membership value of $1 - I(u_i, u_{i+1})$.

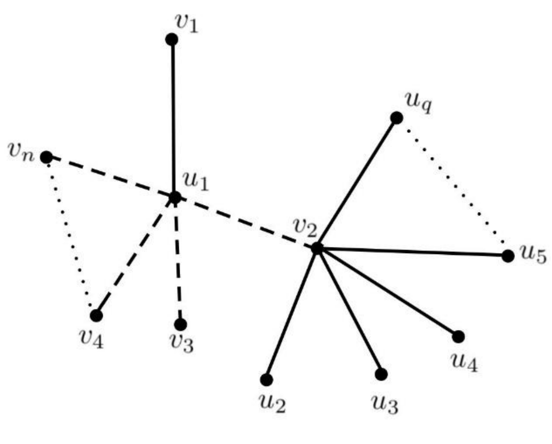

Figure 2. An arbitrary graph.

Subcase 3.1.2.2. *Suppose all the neighboring vertices of v_2 are colored.*
If an edge (v_2, u_k) incident to v_2 is strong, then v_2 cannot be colored with the color of u_k. In other words, if the color of u_k is $(c_k, f(c_k))$, then v_2 cannot be colored with any fuzzy color of c_k. Suppose, v_2 has some weak incident edges, say $(v_2, u_i), i = 1, 2, \ldots, q$. Without loss of

generality assume that, the color of u_i is $(x_i, f(x_i)), i = 1,2,\ldots,q$, where $f(x_i), i = 1,2,\ldots,q$ are membership values of the color $x_i, i = 1,2,\ldots,q$ and $x_i, i = 1,2,\ldots,q$ may be different or same.

To determine the color of v_2, calculate the strength of each weak incident edge and let $M = \max\{1 - I(v_2,u_i), i = 1,2,\ldots,q\}$. Assume that, M is attained for the edge (v_2,u_p). i.e., $M = 1 - I(v_2,u_p)$. If the color of u_p is $(x_p, f(x_p))$ then v_2 will receive the fuzzy color (x_p, M).

If any neighboring vertices $u_i, i = 1,2,\ldots,q$, except u_p, have a basic color, then the basic color must be dilute into a fuzzy color corresponding to that basic color with a membership value $1 - I(v_2,u_i)$.

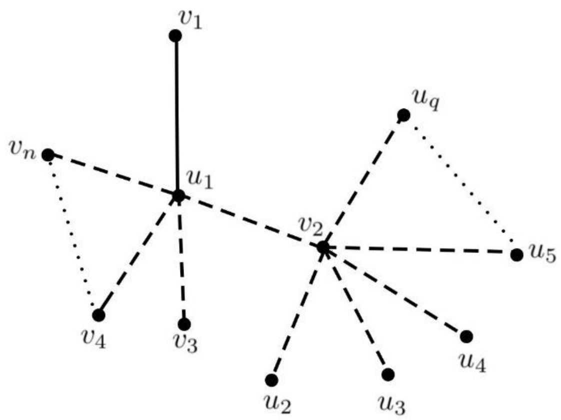

Figure 3. An arbitrary graph.

Subcase 3.1.2.3. *Suppose some of the neighboring vertices of v_2 are colored.*

The neighboring vertices that are not colored do not affect the coloring of v_2. Instead, the neighboring vertices that are colored will be considered for the coloring of v_2. The process of coloring v_2 is similar to the Subcase 3.1.2.2.

After the coloring of v_2, all other vertices are to be colored in a similar manner.

Case.3.1.3. *If all the edges are weak in a fuzzy graph.*

In this case, choose one vertex v_k and color it with any basic color, say $(c_k, 1)$. All other vertices will then receive some fuzzy colors corresponding to the color $(c_k, 1)$. The membership values of the fuzzy colors are calculated using the method described in Subcase 3.1.2.2. Afterwards, color the neighboring vertices of v_k first, and then apply the same coloring method to all vertices.

Note : If a fuzzy graph is disconnected, i.e., if the fuzzy graph has more than one component, each component is colored using the method described above.

Corollary 3.1.1. *If G is a fuzzy graph with components $G[V_1], G[V_2], \ldots, G[v_\omega]$, then $\chi_f(G) = \max\{\chi_f\ (G[V_i]) : 1 \le i \le k\}$.*

Proposition 3.1.1. *If G is a connected fuzzy graph with blocks $B_1, B_2, \ldots, B_k$, then $\chi_f(G) = max\{\chi_f(B_i) : 1 \le i \le k\}$.*

Proof. Let $B_1, B_2, \ldots, B_k$ be the blocks. Consider two blocks B_1 and B_2.

Let $A_1 = B_1 \cup B_2$. Then A_1 be a connected fuzzy graph with blocks B_1 and B_2. Therefore, by Theorem 2.8 we have,

$$\chi_f(A_1) = \chi_f(B_1 \cup B_2) \\ = max\{\chi_f(B_1), \chi_f(B_2)\}.$$

Now consider the fuzzy graph A_1 and the block B_3. Let $A_2 = A_1 \cup B_3$. Then A_2 be a connected fuzzy graph with blocks B_1, B_2 and B_3. Therefore, by Theorem 2.8 we have,

$$\chi_f(A_2) = \chi_f(A_1 \cup B_3) \\ = max\{\chi_f(A_1), \chi_f(B_3)\} \\ = max\{\chi_f(B_1), \chi_f(B_2), \chi_f(B_3)\}.$$

Extend the joining process up to B_k. Then let $A_{k-1} = A_{k-2} \cup B_k = G$. Then A_{k-1} be a connected fuzzy graph with blocks $B_1, B_2, \dots, B_k$. Therefore, by Theorem 2.8 we have

$$\chi_f(A_{k-1}) = \chi_f(G) \\ = \chi_f(A_{k-2} \cup B_k) \\ = max\{\chi_f(A_{k-2}), \chi_f(B_k)\} \\ = max\{\chi_f(B_1), \chi_f(B_2), \dots, \chi_f(B_k)\} \\ = max\{\chi_f(B_i) : 1 \le i \le k\}.$$

$$\therefore \chi_f(G) = max\{\chi_f(B_i) : 1 \le i \le k\}$$

□

3.2. Examples for Fuzzy Coloring and the Chromatic Number of Fuzzy Graphs

Some examples of the chromatic number of certain fuzzy graphs are illustrated below. Consider the fuzzy cycle C_3. If all the edges are weak in C_3, only one basic color is needed to color all the vertices. i.e., the vertex v_1 is colored with $(R, 1)$, and the remaining vertices are colored with different fuzzy colors, $(R, 0.a)$ and $(R, 0.b)$, where a, b are positive integers. Therefore, the chromatic number of this fuzzy cycle is one. If all the edges are strong in C_3, then three basic colors are required to color the vertices. i.e., the vertices v_1, v_2, v_3 are colored with different basic colors $(R, 1), (G, 1)$ and $(Y, 1)$, respectively. Therefore, the chromatic number of this fuzzy cycle is three, which is equal to the chromatic number of the underlying crisp graph.

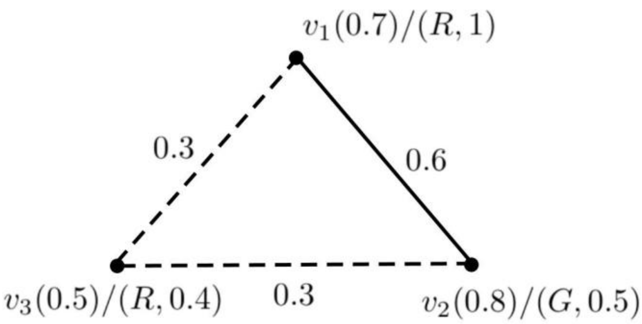

Figure 4. $\chi_f(C_3) = 2$.

Note : Here dotted lines represent weak edges, and plain lines represent strong edges.

Now consider the above fuzzy cycle (Figure 4). Here two edges incident to exactly one vertex (say v_3) are weak, and the remaining edge is strong. First, color the vertex v_1 with a basic color $(R, 1)$. Since the edge (v_1, v_3) is weak, the vertex v_3 will have a fuzzy color $(R, 1 - s)$, where s is the strength of the edge(v_1, v_3), i.e.,$(R, 0.5)$ in this case. Now consider the vertex v_2 for coloring. Since the edge (v_2, v_3) is weak, and the edge (v_1, v_2) is strong, v_2 will have a fuzzy color corresponding to different basic color, say green, with a membership value $1 - I(v_2, v_3)$, i.e.,$(G, 0.5)$ in this case. Hence, the chromatic number is two.

Remark : For C_3, we do have three chromatic numbers based on the membership values of the given fuzzy graph.

4. The Chromatic Number of Certain Families of Fuzzy Graphs

In this section, the chromatic numbers of certain families of fuzzy graphs, such as path, cycle, star, wheel and complete graphs are found by using fuzzy colors based on the strength of an edge incident to a vertex in the fuzzy graph G.

4.1. The Chromatic Number of a Fuzzy Path

Lemma 4.1.1. *Let P_n be a fuzzy path of length n. If all edges are weak in P_n, then* $\chi_f(P_n) = 1$.

Lemma 4.1.2. *Let P_n be a fuzzy path of length n. If all the edges are strong in P_n, then* $\chi_f(P_n)= 2$.*(By Theorem 2.2).*

Theorem 4.1.1. *Let P_n be a fuzzy path of length n. If atleast one edge is strong in P_n, then* $\chi_f(P_n) = 2$. *(By the procedure 3.1).*

Property 4.1.1. Let P_{2n-1} be a fuzzy path of length $2n - 1$. Then, by proper fuzzy coloring, the following statements are true :

(i) Suppose *even* number of weak edges, along with strong edges, are distributed in any sequence (except alternative distribution) in P_{2n-1}. Then, end vertices of P_{2n-1} can either have different basic colors, or one vertex can have a basic color and the other can have a different fuzzy color.

(ii) Suppose *odd* number of weak edges, along with strong edges, are distributed in any sequence (except alternative distribution) in P_{2n-1}. Then, end vertices of P_{2n-1} can either have same basic colors, or one vertex can have a basic color and the other can have a fuzzy color corresponding to the same basic color.

(iii) Suppose *even* number of weak edges, along with strong edges, are alternatively distributed in P_{2n-1}. Then, end vertices of P_{2n-1} can have different basic colors.

(iv) Suppose *odd* number of weak edges, along with strong edges, are alternatively distributed in P_{2n-1}. Then, one end vertices of P_{2n-1} can have a basic color and the other can have a fuzzy color corresponding to the same basic color.

Property 4.1.2. Let P_{2n} be a fuzzy path of length $2n$. Then, by proper fuzzy coloring, the following statements are true :

(i) Suppose *even* number of weak edges, along with strong edges, are distributed in any sequence in P_{2n} . Then, end vertices of P_{2n} can either have same basic colors, or one vertex can have a basic color and the other can have a fuzzy color corresponding to the same basic color.

(ii) Suppose *odd* number of weak edges, along with strong edges, are distributed in any sequence in P_{2n}. Then, end vertices of P_{2n} can either have different basic colors, or one vertex can have a basic color and the other can have a different fuzzy color.

(iii) Suppose *even* number of weak edges, along with strong edges, are alternatively distributed in P_{2n}. Then, end vertices of P_{2n} can either have same basic colors or one vertex can have a basic color and the other can have a fuzzy color corresponding to the same basic color.

(iv) Suppose *odd* number of weak edges, along with strong edges, are alternatively distributed in P_{2n}. Then, end vertices of P_{2n} can either have different basic colors or one vertex can have a basic color and the other can have a different fuzzy color.

4.2. The Chromatic Number of a Fuzzy Cycle

Lemma 4.2.1. *Let C_n be a fuzzy cycle of length n. If all edges are weak in C_n, then* $\chi_f(C_n) = 1$.

Lemma 4.2.2. *Let C_n be a fuzzy cycle of length n. If all the edges are strong in C_n, then (by Theorem 2.3)*

$$\chi_f(C_n) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Theorem 4.2.1. *Let C_n be a fuzzy cycle of length n. If weak and strong edges are distributed in any sequence in C_n, then*

$$\chi_f(C_n) = \begin{cases} 3 & if\ \frac{n}{2}\ number\ of\ strong\ and\ weak\ edges\ are\ alternatively \\ & distributed\ in\ C_n,\ where\ n(\geq 6) is\ even, \\ 2 & otherwise. \end{cases}$$

Proof. **Case 1 :** Suppose $\frac{n}{2}$ number of strong and weak edges are alternatively distributed in C_n, $n\ (\geq\ 6)$ is even.

Let $C_{2n}: v_1e_1 \ldots e_{2n-1}v_{2n}e_{2n}v_1$ be a fuzzy cycle of length $2n, n \geq 3$ and $C_{2n} = P_{2n-1} + e_{2n}$, where $P_{2n-1}: v_1e_1 \ldots e_{2n-1}v_{2n}$ be a fuzzy path of length $2n-1$. Then by Theorem 4.1.1, $\chi_f(P_{2n-1}) = 2$ and by Property 4.1.1, end vertices of P_{2n-1} can have the following coloring possibilities :

Subcase 1.1 : If $e_1 \in P_{2n-1}$ is strong, then $e_{2n} \in C_{2n}$ will be weak. Then v_1 & v_{2n} will have different basic colors, say $(c_1, 1)$ and $(c_2, 1)$ respectively. Since the edge e_{2n} is weak, v_{2n} will receive a third color, say $(c_3, 1)$ and hence $v_1$will receive a fuzzy color of $(c_3, 1)$, say $(c_3, f(c_3))$ with membership value $1 - I(v_1, v_{2n})$. $\therefore$ $\chi_f(C_n) = 3$.

Subcase 1.2 : If $e_1 \in P_{2n-1}$ is weak, then $e_{2n} \in C_{2n}$will be strong. Then $v_1$has basic color, say $(c_1, 1)$ and v_{2n} has fuzzy color, say $(c_1, f(c_1))$. Since the edge e_{2n} is strong, v_{2n}will receive a fuzzy color of third color, say $(c_3, f(c_3))$. $\therefore$ $\chi_f(C_n) = 3$.

Case 2 : Suppose $\left\lfloor \frac{n}{2} \right\rfloor$ number of strong and weak edges are alternatively distributed in $C_n, n\ (\geq 3)$ is odd.

Let $C_{2n+1}: v_1e_1 \ldots e_{2n}v_{2n+1}e_{2n+1}v_1$ be a fuzzy cycle of length $2n+1, n \geq 1$ and $C_{2n+1} = P_{2n} + e_{2n+1}$, where $P_{2n+1}: v_1e_1 \ldots e_{2n}v_{2n+1}$ be a fuzzy path of length $2n$.

Then by Theorem 4.1.1, $\chi_f(P_{2n}) = 2$, and by Property 4.1.2, end vertices of P_{2n} can have the following coloring possibilities :

Subcase 2.1 : Suppose there is an even number of weak edges in P_{2n}, and $v_1$has basic color, say $(c_1, 1)$, while $v_{(2n+1)}$ has fuzzy color, say $(c_1, f(c_1))$. If $e_1 \in P_{2n}$ is strong, then $e_{2n+1} \in C_{2n+1}$will be strong. Since the edge e_{2n+1} is strong, v_{2n+1} will receive a fuzzy color of second color, say $(c_2, f(c_2))$. $\therefore$ $\chi_f(C_n) = 2$.

Subcase 2.2 : Suppose there is an even number of weak edges in P_{2n}, and v_1 & v_{2n+1} have same basic colors, say$(c_1, 1)$. If $e_1 \in P_{2n}$ is weak, then $e_{2n+1} \in C_{2n+1}$will be weak. Since the edge e_{2n+1} is weak, color of $v_1$will diluted into a fuzzy color, say $(c_1, f(c_1))$, with membership value $1 - I(v_1, v_{2n+1})$. $\therefore$ $\chi_f(C_n) = 2$.

Subcase 2.3 : Suppose there is an odd number of weak edges in P_{2n}, and v_1 has basic color, say $(c_1, 1)$, while v_{2n+1}has different fuzzy color, say$(c_2, f(c_2))$. If $e_1 \in P_{2n}$ is strong, then $e_{2n+1} \in C_{2n+1}$ will be strong. Since the edge e_{2n+1} is strong, $v_1$and v_{2n+1} can have same colors as mentioned. $\therefore\ \chi_f(C_n) = 2$.

Subcase 2.4 : Suppose there is an odd number of weak edges in P_{2n}, and v_1 & v_{2n+1} have different basic colors, say $(c_1, 1)$ and $(c_2, 1)$ respectively. If $e_1 \in P_{2n}$ is weak, then $e_{2n+1} \in C_{2n+1}$ will be weak. Since e_{2n+1} is weak, v_1 will receive a fuzzy color of second color, say $\left(c_2, f(c_2)\right)$, with membership value $1 - I(v_1, v_{2n+1})$.
$\therefore\ \chi_f(C_n) = 2$.

Case 3 : Suppose weak and strong edges are distributed in any sequence (except alternative distribution) in C_n.
Let $C_n : v_1 e_1 \dots v_n e_n v_1$ be a fuzzy cycle of length n and $C_n = P_{n-1} + e_n$, where $P_{n-1} : v_1 e_1 \dots e_{n-1} v_n$ be a fuzzy path of length $n - 1$. By Theorem 4.1.1, $\chi_f(P_{n-1}) = 2$ and consider the edge $e_n = (v_n, v_1) \in C_n$.
Suppose the edge e_n is weak. By Property 4.1.1 or Property 4.1.2, end vertices of P_{n-1} can have different coloring possibilities and then by the procedure of fuzzy coloring, $\chi_f(C_n) = 2$. (coloring procedure will be same as above).
Suppose the edge e_n is strong. By Property 4.1.1 or Property 4.1.2, end vertices of P_{n-1} can have different coloring possibilities and then by the procedure of fuzzy coloring, $\chi_f(C_n) = 2$. (coloring procedure will be same as above). □

Note :

(i) When $n = 2$ in $C_n, \chi_f(C_2) = 2$.

(ii) When $n = 4$ in $C_n, \chi_f(C_4) = 2$.

4.3. The Chromatic Number of a Fuzzy Star

Lemma 4.3.1. *Let S_n be a fuzzy star. If all edges are weak in S_n, then $\chi_f(S_n) = 1$.*

Lemma 4.3.2. *Let S_n be a fuzzy star. If all the edges are strong in S_n, then $\chi_f(S_n) = 2$. (By Theorem 2.4).*

Theorem 4.3.1. *Let S_n be a fuzzy star. If atleast one edge is strong in S_n, then $\chi_f(S_n) = 2$. (By the procedure 3.1).*

4.4. The Chromatic Number of a Fuzzy Wheel

Lemma 4.4.1. *Let $W_n, n \geq 3$ be a fuzzy wheel. If all edges are weak in W_n, then $\chi_f(W_n) = 1$.*

Lemma 4.4.2. *Let $W_n, n \geq 3$ be a fuzzy wheel. If all the edges are strong in W_n, then (by Theorem 2.5)*

$$\chi_f(W_n) = \begin{cases} 4 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Theorem 4.4.1. *Let $W_n, n \geq 3$ be a fuzzy wheel and $W_n = S_{n-1} \oplus C_{n-1}$, where S_{n-1} be a fuzzy star and C_{n-1} be a fuzzy cycle of length $n - 1$. If weak and strong edges are distributed in any sequence in W_n, then*

$$\chi_f(W_n) = \begin{cases} 4 & if \ \frac{n-1}{2} \ number \ of \ strong \ and \ weak \ edges \ are \ alternatively \\ & distributed \ in \ C_{n-1}, where \ n(\geq 7) is \ odd, \\ 3 & otherwise. \end{cases}$$

Proof. Let W_n be a fuzzy wheel and $W_n = S_{n-1} \oplus C_{n-1}$, where S_{n-1} be a fuzzy star and C_{n-1} be a fuzzy cycle of length $n-1$. Then by Theorem 4.3.1 we have, $\chi_f(S_{n-1}) = 2$ and by Theorem 4.2.1 we have,

$$\chi_f(C_{n-1}) = \begin{cases} 3 & if \ \frac{n-1}{2} \ number \ of \ strong \ and \ weak \ edges \ are \ alternatively \\ & distributed \ in \ C_{n-1}, where \ n(\geq 7) is \ odd, \\ 2 & otherwise. \end{cases}$$

We know that by Theorem 2.8,

$$\chi_f(W_n) = \max\{\chi_f(S_{n-1}), \chi_f(C_{n-1})\} + 1. \quad (1)$$

Case 1 : Suppose $\frac{n-1}{2}$ number of strong and weak edges are alternatively distributed in $C_{n-1}, n(\geq 7)$ is odd, then using the result (1), we have

$$\chi_f(W_n) = \max\{2, 3\} + 1 = 4.$$

Case 2 : Suppose Case 1 is not occur, then using the result (1), we have

$$\chi_f(W_n) = \max\{2, 2\} + 1 = 3.$$

□

Note :

(i) When $n = 3$ in $W_n, \chi_f(W_3) = 3$.

(ii) When $n = 5$ in $W_n, \chi_f(W_5) = 3$.

Note : Let W_n be a fuzzy wheel and $W_n = S_{n-1} \oplus C_{n-1}$, where S_{n-1} be a fuzzy star and C_{n-1} be a fuzzy cycle of length $n-1$. The edges of $S_{n-1} \in W_n$ are referred to as the inner edges of W_n, while the edges of $C_{n-1} \in W_n$ are referred to as the outer edges of W_n.

Corollary 4.4.1.1. *Let W_n be a fuzzy wheel and $W_n = S_{n-1} \oplus C_{n-1}$, where S_{n-1} be a fuzzy star and C_{n-1} be a fuzzy cycle of length $n-1$. If weak and strong edges are distributed in any sequence in $C_{n-1} \in W_n$, or if all the outer edges of W_n are strong and all the inner edges of W_n are weak,, then $\chi_f(W_n) = \chi_f(C_{n-1})$.*

Corollary 4.4.1.2. *Let W_n be a fuzzy wheel and $W_n = S_{n-1} \oplus C_{n-1}$, where S_{n-1} be a fuzzy star and C_{n-1} be a fuzzy cycle of length $n-1$. If weak and strong edges are distributed in any sequence in $S_{n-1} \in W_n$, or if all the inner edges of W_n are strong and all the outer edges of W_n are weak, then $\chi_f(W_n) = \chi_f(S_{n-1})$.*

Corollary 4.4.1.3. *Let W_n be a fuzzy wheel and $W_n = S_{n-1} \oplus C_{n-1}$, where S_{n-1} be a fuzzy star and C_{n-1} be a fuzzy cycle, having $\chi_f(C_{n-1}) = 2$. Suppose all the outer edges of W_n are strong and the inner edges of W_n, which connect vertices with the same color, are weak. Then, $\chi_f(W_n) = 2$.*

4.5. The Chromatic Number of a Fuzzy Complete Graph

Lemma 4.5.1. *Let K_n be a complete fuzzy graph. If all edges are weak in K_n, then* $\chi_f(K_n) = 1$.

Lemma 4.5.2. *Let K_n be a complete fuzzy graph. If all the edges are strong in K_n, then $\chi_f(K_n) = n$. (By Theorem 2.6).*

Note :

(i) $K_{2n+1} = \oplus\, nC_{2n+1}$ (by Theorem 2.7). In K_{2n+1}, the edges of the outer cycle $C_{2n+1} : v_1 e_1 v_2 \ldots v_{2n+1} e_{2n+1} v_1$ are referred to as the outer edges of K_{2n+1}, and the edges of other cycles in K_{2n+1} are referred to as the inner edges of K_{2n+1}.

(ii) $K_{2n} = C_{2n} \oplus nP_1$ (by Theorem 2.7). In K_{2n}, the edges of the outer cycle $C_{2n} : v_1 e_1 v_2 \ldots v_{2n} e_{2n} v_1$ are referred to as the outer edges of K_{2n}, while the edges of other cycles in K_{2n} and the edges of a perfect matching in K_{2n} are referred to as the inner edges of K_{2n}.

Theorem 4.5.1. *Let K_n be a complete fuzzy graph. If weak and strong edges are distributed in any sequence in K_n, then, $\chi_f(K_n) = n - \lfloor \frac{m}{2} \rfloor$, where m $(m < n)$be the number of vertices having atleast one weak incident edge.*

Proof. Let K_n be a complete fuzzy graph and $C_n : v_1 e_1 v_2 \ldots v_n e_n v_1$ be the outer cycle of K_n.

Case 1 : Let $e_1 e_2 \ldots e_{m-1} \in C_n$ $(m < n)$ be the weak edges in K_n and the remaining edges in K_n are strong.

Color the vertex $v_1 \in C_n$ with a basic color, say $(c_1, 1)$. Since the edge $e_1 \in C_n$ is weak, vertex $v_2 \in C_n$ will receive a fuzzy color of the same basic color c_1, say $(c_1, f(c_1))$ with membership value $1 - I(v_1, v_2)$. As the inner edge (v_1, v_3) of K_n is strong and since the edge $(v_2, v_3) = e_2 \in C_n$ is weak, vertex $v_3 \in C_n$ will receive a fuzzy color of different basic color c_2, say $(c_2, f(c_2))$ with membership value $1 - I(v_2, v_3)$. Consequently, vertex $v_4 \in C_n$ will also receive a fuzzy color of the same color c_2, say $(c_2, f(c_2))$ with membership value $1 - I(v_3, v_4)$. Extend the coloring process up to m vertices (by the procedure 3.1).

Subcase 1.1 : m is even.

The end vertices of each edges $e_i, i = 1, 3, \ldots, e_{m-1}$ will receive fuzzy colors of same basic colors, say $(c_j, f(c_j)), j = 1, 2, \ldots, \frac{m}{2}$. Therefore, $\frac{m}{2}$ different colors are used to color the m vertices. Since each $n - m$ vertices are strongly adjacent (two vertices are connected by a strong edge) to the remaining $n - 1$ vertices of K_n, they will receive $n - m$ different colors and will not receive any colors that have already been used(by the procedure 3.1). Hence,

$$\chi_f(K_n) = (n - m) + \frac{m}{2}$$
$$= \frac{2n - m}{2}$$
$$\equiv n - \left\lfloor \frac{m}{2} \right\rfloor.$$

Subcase 1.2 : m is odd.
The end vertices of each edges $e_i, i = 1,3,\ldots,e_{m-2}$ will receive fuzzy colors of same basic colors, say $(c_j, f(c_j)), j = 1,2,\ldots,\lfloor\frac{m}{2}\rfloor + 1$. Therefore, $\lfloor\frac{m}{2}\rfloor + 1$ different colors are used to color the m vertices. Since each $n-m$ vertices are strongly adjacent (two vertices are connected by a strong edge) to the remaining $n-1$ vertices of K_n, they will receive $n-m$ different colors and will not receive any colors that have already been used(by the procedure 3.1). Hence,

$$\chi_f(K_n) = (n-m) + \left\lfloor\frac{m}{2}\right\rfloor + 1$$
$$\equiv n - \left\lfloor\frac{m}{2}\right\rfloor.$$

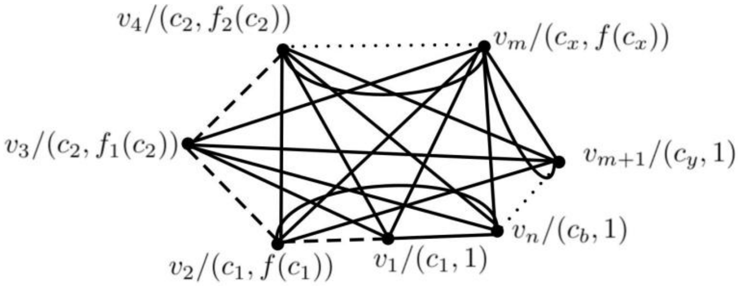

Figure 5. Complete Fuzzy Graph.

Case 2 : Let all the inner edges and some outer edges in K_n be strong, while the remaining outer edges in K_n are weak.

Case 3 : Let some inner and outer edges in K_n be weak, while the remaining edges in K_n are strong.
In both cases (case 2 & case 3), color the end vertices of weak edges first. Then, color the remaining vertices according to the procedure described in the previous case. Then

$$\chi_f(K_n) = n - \left\lfloor\frac{m}{2}\right\rfloor. \qquad \square$$

5. *Application of Fuzzy Coloring*

In this study, we analysed the literacy rates of various states in India and explored the relationships between these states with the goal of improving literacy. We are working on finding a solution for *how many states can offer high-quality education to improve their literacy rates by implementing educational quotas for other states*. This concept is demonstrated using fuzzy coloring. For that, we created a fuzzy graph model, taking into account some states in India (Kerala, Tamil Nadu, Karnataka, Andhra Pradesh, Telangana, Maharashtra, Lakshadweep) as vertices, with edges connecting them, if they have mutual cooperation to improve the literacy of the states. The membership values of the vertices represent the literacy rates of the states (with respect to educational literacy, computer literacy, media literacy, linguistic literacy, health literacy, critical literacy, statistical literacy, etc). The membership value of an edge indicates the level of relationship between the states aimed at improving literacy (by offering the educational quotas for other states, empowering communities such as those focused on rural literacy development, supporting disabled communities, and promoting women's literacy initiatives, etc). i.e., Here we are considering the relationships between some states in India that offer educational quotas for other states to help improve their literacy rates.

Let the states Kerala, Tamil Nadu, Karnataka, Andhra Pradesh, Telangana, Maharashtra, and Lakshadweep are considered as vertices and are denoted by $A, B, C, D, E, F,$ and G respectively (refer Figure 6). The literacy rates of these states will be the membership values of each vertex, which are $0.8, 0.7, 0.5, 0.7, 0.6, 0.6,$ and 0.5 respectively. Two vertices are connected, if and only if they offer educational quotas to each other.

Let the edges $(A, B), (A, C), (A, F), (B, C), (B, D), (B, E), (C, D), (C, F), (D, E), (D, F), (E, F)$ represent the relation between states that offer educational quotas to each other (refer Figure 7). The membership values of each edge, which denote the strength of the relationship between the states in providing educational quotas, are $0.7, 0.2, 0.6, 0.3, 0.7, 0.4, 0.5, 0.4, 0.6, 0.3,$ and 0.5 respectively. Moreover, the edges (A, B), (A, F), $(B, C), (B, D), (C, D), (D, E), (E, F)$ are strong, representing strong relations between the states and the edges $(A, C), (B, E), (C, F), (D, F)$ are weak, indicating weaker relations between the states compared to the other relations. Since Lakshadweep has no relation with other states in providing the educational quota, the vertex G is disconnected. So the graphical representation of this model problem contains two components $G[V_1]$ and $G[V_2]$ (refer Figure 7). Now, by calculating the chromatic number of this fuzzy graph, we can determine how many states offer high-quality education to raise their literacy rates by implementing educational quotas for other states.

Figure 6. Let A,B,C,D,E,G, and F are the few sates of India.

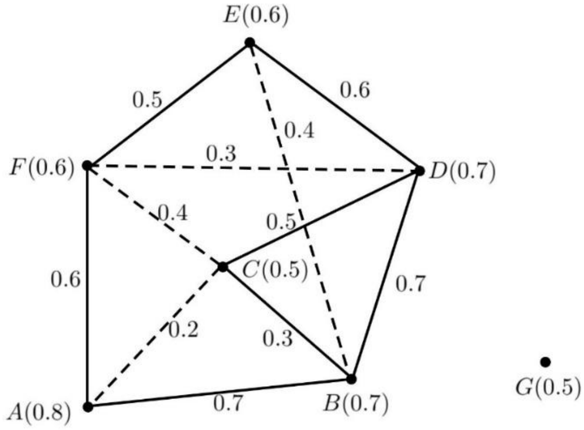

Figure 7. A fuzzy graph representation of Figure 6.

First, consider the connected component $G[V_1]$ for coloring. It has both strong and weak edges. So we can color the graph by Case 3.1.1 & Case 3.1.2 of fuzzy coloring of fuzzy graph. First color the vertex A with arbitrary basic color, say $(G, 1)$. As the edge (A, B) is strong, the vertex B will receive a different basic color, say $(R, 1)$. Since the edge (A, C) is weak, the vertex C will receive a fuzzy color corresponding to the color of the vertex A, say $(G, 0.6)$. Similarly we can color the vertex E with a fuzzy color $(R, 0.4)$. Since the edges (B, D) and (C, D) are strong, the vertex D will receive another basic color say, $(Y, 1)$. Now consider the vertex F for coloring. Coloring of the vertex F depends on the weak incident edges (C, F) and (D, F). Then vertex F is colored with a fuzzy color that corresponds to one of the colors of the two vertices. The membership value of the fuzzy color is determined by calculating the strengths of each weak edges. Since the edge (D, F) has minimum strength, F will receive a fuzzy color $(Y, 0.5)$. Now consider the second component $G[V_2]$ for coloring. Since the component $G[V_2]$ is a trivial graph, the vertex G can be colored with any color used in the coloring of the component $G[V_1]$. Hence, the vertex G is colored with a basic color $(R, 1)$. Then by Corollary 3.1.1, $\chi_f(G) = max\{3, 1\} = 3$. i.e., the graph G is colored with only three colors namely Green, Red and Yellow.

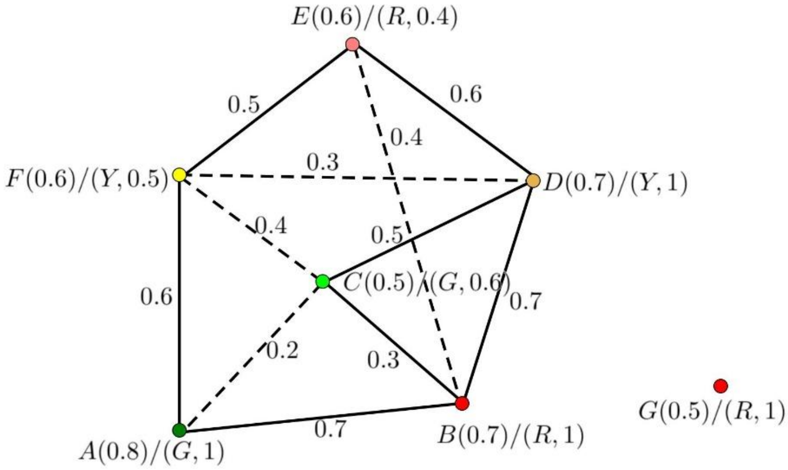

Figure 8. Perfect fuzzy coloring of the fuzzy graph G.

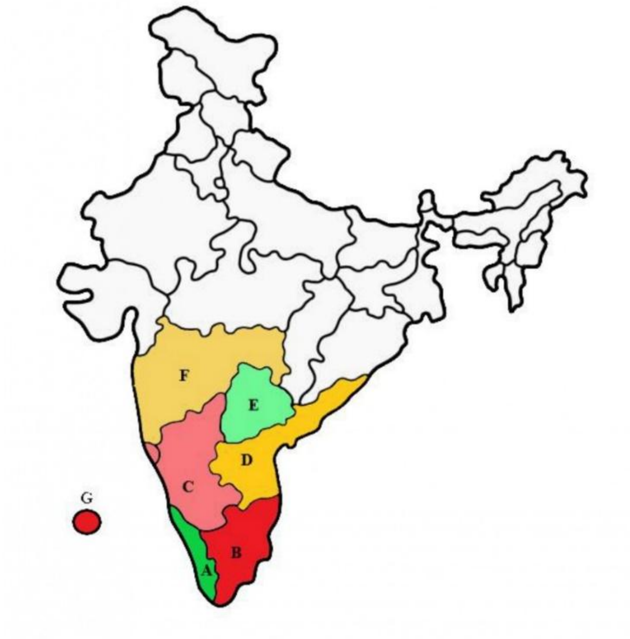

Figure 9. Representation of fuzzy coloring of few state of India.

In this study, the chromatic number of G is three. Therefore, three states can offer high-quality education to improve their literacy rates by implementing educational quotas for other states. Therefore, to achieve high-quality education, a state must either forge strong bonds with any of these three states or strengthen existing relationships with other states by enhancing educational quotas.

6. Conclusion

In this paper, we discussed the key concepts essential for fuzzy coloring and introduced an improved procedure for a fuzzy coloring of fuzzy graphs. The chromatic numbers of certain families of fuzzy graphs are found by using fuzzy colors based on the strength of an edge incident to a vertex.
In our further study, it is proposed to work on fuzzy coloring of product graphs and we analyse the concept of chromatic number on it.

7. References

[1] Baghel, M., Agrawal, S., & Silakari, S. (2013). Recent trends and developments in graph coloring. In *Proceedings of the International Conference on Frontiers of Intelligent Computing: Theory and Applications (FICTA)* (pp. 431-439). Springer Berlin Heidelberg.

[2] Samanta, S., Pramanik, T., & Pal, M. (2016). Fuzzy colouring of fuzzy graphs. *Afrika Matematika*, *27*, 37-50.

[3] Munoz, S., Ortuno, M. T., Ramírez, J., & Yanez, J. (2005). Coloring fuzzy graphs. *Omega*, *33*(3), 211-221.

[4] Eslahchi, C., & Onagh, B. N. (2006). Vertex-strength of fuzzy graphs. *International Journal of Mathematics and Mathematical Sciences*, *2006*(1), 043614.

[5] Chartrand, G., & Zhang, P. (2019). *Chromatic graph theory*. Chapman and Hall/CRC.

[6] Jayapriya, B., Kamali, R., Ravithammal. (2021). *Coloring in Various Graphs*. Gedrag & Organisatie Review, 34 (3):244-250.

[7] Karunambigai, M. G., & Muthusamy, A. (2006). On Resolvable Multipartite G-Designs II. *Graphs and Combinatorics*, *22*, 59-67.

[8] Gong, Z., & Zhang, C. (2023). *Adjacent Vertex Distinguishing Coloring of Fuzzy Graphs.* Mathematics, 11(10), 2233.

[9] Mathew, S., Mordeson, J. N., & Malik, D. S. (2018). *Fuzzy graph theory* (Vol. 363). Berlin, Germany: Springer International Publishing.

[10] Gong, Z., & Zhang, J. (2022). *Chromatic Number of Fuzzy Graphs: Operations, Fuzzy Graph Coloring, and Applications.* Axioms, 11(12), 697.

[11] Rehmani, S., & Sunitha, M. S. (2018, November). *Forcing geodesic number of a fuzzy graph.* In Journal of Physics: Conference Series (Vol. 1132, No. 1, p. 012062). IOP Publishing.

[12] Pal, M., Samanta, S., & Ghorai, G. (2020). *Modern trends in fuzzy graph theory* (pp. 7-93). Berlin: Springer.

[13] Sudha, T., & Jayalalitha, G. (2021). *Application of fuzzy network graph in hospital.* Adv. Appl. Math. Sci, 20(9), 1823-1829.